Freshman Physics Laboratory
Physics 3 Course

Experiments Notes
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Virgínio de Oliveira Sannibale, Kenneth G. Libbrecht

Caltech Physics Department, MS 103-33
1200 E. California Blvd.
91125 Pasadena, CA
http://www.ligo.caltech.edu/~vsanni/ph3/
Contents

1 Data Analysis 7

2 The Maxwell Top 11
  2.1 Introduction ............................................. 11
  2.2 Some Relevant Examples ................................. 12
      2.2.1 Angular Acceleration under a Constant Torque .... 12
      2.2.2 Top Suspended with a Torsional Rod .......... 13
      2.2.3 Precession of the Top .......................... 14
  2.3 Experimental setup ................................... 16
      2.3.1 Care and Use of the Experimental Apparatus .... 16
  2.4 First Laboratory Week ................................. 18
      2.4.1 Indirect Measurement of the Moment of Inertia Applying a Constant Torque ........ 18
      2.4.2 Indirect Measurement of the Moment of Inertia Using a Torsional Pendulum .... 18
      2.4.3 Propaedeutic Problems ............................ 19
      2.4.4 Procedure (Top’s Moment of Inertia Measurements) 19
  2.5 Second Laboratory Week ............................... 21
      2.5.1 Propaedeutic Problems ............................ 22
      2.5.2 Procedure (Precession Period Measurement) .... 23

3 The Inverted Pendulum 25
  3.1 Introduction ............................................. 25
  3.2 Modeling the Inverted Pendulum (IP) .................. 26
      3.2.1 The Simple Harmonic Oscillator .................. 26
      3.2.2 The Simple Pendulum ............................. 26
      3.2.3 The Simple Inverted Pendulum ................. 27
      3.2.4 A Better Model of the Inverted Pendulum .... 29
3.3 The Damped Harmonic Oscillator ........................................ 30
3.4 The Driven Harmonic Oscillator ....................................... 31
3.5 The Transfer Function ..................................................... 33
3.6 The Inverted Pendulum Test Bench ..................................... 34
  3.6.1 Care and Use of the Apparatus .................................... 34
3.7 The Lab - First Week ..................................................... 34
  3.7.1 Pre-Lab Problems ...................................................... 34
  3.7.2 In-Lab Exercises ........................................................ 35
    3.7.2.1 Getting Started .................................................. 35
    3.7.2.2 The IP Leg ........................................................ 36
    3.7.2.3 Resonant Frequency versus Load .............................. 36
3.8 The Lab - Second Week ................................................... 39
  3.8.1 Pre-Lab Problems ...................................................... 39
  3.8.2 In-Lab Exercises ........................................................ 40
    3.8.2.1 Inverted Pendulum Loss Angle ................................. 40
    3.8.2.2 The Transfer Function ........................................ 40
3.9 EndNote: Using Complex Functions to Solve Real Equations .... 41

4 Direct Current Network Theory ............................................. 43
  4.1 Electronic Networks ..................................................... 43
    4.1.1 Network Definitions .............................................. 43
    4.1.2 Series and Parallel .............................................. 44
    4.1.3 Active and Passive Components ................................ 45
  4.2 Kirchhoff’s Laws ......................................................... 45
  4.3 Resistors (Ohm’s Law) ................................................... 45
    4.3.1 Resistors in Series ................................................ 46
    4.3.2 Resistors in Parallel ............................................. 47
  4.4 Capacitors ................................................................. 48
    4.4.1 Capacitors in Parallel ............................................ 48
    4.4.2 Capacitors in Series .............................................. 49
  4.5 Ideal and Real Sources .................................................. 50
    4.5.1 Ideal Voltage Source ............................................. 50
    4.5.2 Ideal Current Source ............................................. 50
  4.6 The Semiconductor Junction (Diode) ................................ 51
  4.7 Equivalent Networks .................................................... 53
    4.7.1 Voltage Divider ..................................................... 53
    4.7.2 Thévenin Theorem ................................................ 54
    4.7.3 Norton Theorem ................................................... 56
## CONTENTS

4.8 Resistor Color Code .................................................. 56
4.9 First Laboratory Week ................................................ 58
  4.9.1 Pre-Laboratory Problems ...................................... 59
  4.9.2 Procedure ......................................................... 60
4.10 Second Laboratory Week ........................................... 62
  4.10.1 Pre-Laboratory Problems ...................................... 62
  4.10.2 Procedure ......................................................... 63

5 Alternating Current Network Theory ............................... 65
  5.1 Symbolic Representation of a Sinusoidal Signals, Phasors ... 65
    5.1.1 Derivative of a Phasor ....................................... 67
    5.1.2 Integral of a Phasor .......................................... 67
  5.2 Current Voltage Equation for Passive Ideal Components with
    Phasors .................................................................. 67
    5.2.1 The Resistor ...................................................... 68
    5.2.2 The Capacitor .................................................... 68
    5.2.3 The Inductor ...................................................... 69
  5.3 The Impedance and Admittance Concept ......................... 70
    5.3.1 Impedance in Parallel and Series ............................ 71
    5.3.2 Ohm’s Law for Sinusoidal Regime ......................... 71
  5.4 Two-port Network ..................................................... 71
    5.4.1 The RC Low-Pass Filter ....................................... 73
    5.4.2 The CR High-Pass Filter ...................................... 74
    5.4.3 The LCR Series Resonant Circuit ......................... 76
  5.5 First Laboratory Week ............................................... 79
    5.5.1 Pre-laboratory Exercises ..................................... 79
    5.5.2 Procedure ......................................................... 80
  5.6 Second Laboratory Week ............................................ 82
    5.6.1 Pre-laboratory Exercises ..................................... 82
    5.6.2 Procedure ......................................................... 83

A Measurements and Significant Figures (Draft) .................. 87
  A.1 Scientific Notation .................................................. 87
  A.2 Significant Figures .................................................. 88
  A.3 Significant Figures in Measurements ............................ 88
    A.3.1 Significant Figures for Direct Measurements ............. 89
      A.3.1.1 Length Measurement Example ........................... 89
    A.3.2 Significant Figures for Statistical Measurements ....... 90
## CONTENTS

A.3.2.1 Electron Mass Example ................................................. 91
A.3.3 Significant Digits in Calculations ................................. 91
A.4 Unit Prefixes ................................................................. 91

B The Vernier ........................................................................... 93
  B.1 Measuring with a Vernier .................................................. 94
  B.2 Probability Density Function Using a Vernier ......................... 95

C The Cathode Ray Tube Oscilloscope ........................................ 97
  C.1 The Cathode Ray Tube Oscilloscope .................................... 97
    C.1.1 The Cathode Ray Tube ............................................... 98
    C.1.2 The Horizontal and Vertical Inputs ................................. 98
    C.1.3 The Time base Generator ........................................... 98
    C.1.4 The Trigger ............................................................. 100
  C.2 Oscilloscope Input Impedance ........................................... 101
  C.3 Oscilloscope Probe ........................................................ 102
    C.3.1 Probe Frequency Compensation .................................... 103
  C.4 Beam Trajectory ............................................................ 106
    C.4.1 CRT Frequency Limit ............................................... 107

D Perturbation of an Electronic Instrument ................................ 109
  D.1 Current Measurement ...................................................... 109
  D.2 Voltage Measurement ...................................................... 111
  D.3 AC Circuit Measurement Perturbation ................................ 112
    D.3.1 Example: Oscilloscope ............................................. 112

E Data Acquisition System “Experimenter” .............................. 115
  E.1 Output Channel Characteristics ....................................... 116
  E.2 ExperTerm Program ......................................................... 117
  E.3 ExperDAQ Program ........................................................ 117
Chapter 1
Data Analysis

Sections 1, 3, and 4 of the notes “Vademecum for Data Analysis Beginners” contain the information necessary to complete test of this chapter. It is indeed mandatory to read those sections before starting answering the questions.

Some of the questions require the use of a computer program named “CurveFit”, which will be explained during the first laboratory class. Sheets to make graphs will be provided during the class.

Data Analysis Questions

1. The thickness $d$ of the base of a cylindrical can is computed by measuring the outside height $H = (97.3 \pm 0.2)$mm and the inside height $h = (97 \pm 1)$mm. What is $d$ and its uncertainty $\sigma_d$? Does the more precise measurement of $H$ affect the uncertainty $\sigma_d$?

2. A computer program gives values for two parameters in the following form:

   $a = 12.37825$ m/s, error on $a =0.0286145$ m/s
   $b = 3.2395e-3$ m, error on $b = 2.7481912e-05$ m

   How do you report these values?

3. A distance $x$ is measured using a ruler of length $l = 300$mm, which is too short for a direct measurement. Using the ruler stepwise nine
times, it is found that the distance \( x \) is equal to 2654mm. If the uncertainty of each measurement is \( \sigma = 1 \text{mm} \), what is the uncertainty \( \sigma_x \) on \( x \) ?

4. We have two measurements of the same quantity obtained using two different techniques

\[
p_1 = (1.231 \pm 0.019) \text{kg/m}^2 \quad \text{and} \quad p_2 = (1.262 \pm 0.012) \text{kg/m}^2 .
\]

Do these two measurements agree?

5. Six measurements are made of the voltage difference across a resistor. The results are as follows:

<table>
<thead>
<tr>
<th>Measurement n.</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Voltage Difference (V)</td>
<td>1.44</td>
<td>1.48</td>
<td>1.47</td>
<td>1.43</td>
<td>1.50</td>
<td>1.47</td>
</tr>
</tbody>
</table>

Compute the uncertainty of each measurement and the uncertainty of mean using matlab commands mean and std, and report the measurements in the proper format.

If the resistance is measured to be \( R = (15.1 \pm 0.1) \Omega \), what is the power \( P = \frac{V^2}{R} \) dissipated by the resistor and its uncertainty \( \sigma_P \)?

6. Let \( I = \frac{1}{2}M(R^2 + r^2) \). What is the equation for \( \sigma_I \), if \( M, R \) and \( r \) are measured quantities? If we require \( \Delta I / I = 0.1\% \), what is the relative error for the measurements of \( M, R \) and \( r \)? (Suppose that \( R = ar, \ a > 1, \) and \( \Delta R = \Delta r \)).

7. The voltage \( V \) along a transmission line is \( V(x) = V_0 \exp(-x/x_a) \), where \( x \) is the position along the line. Find \( x_a \) and its uncertainty \( \sigma_{x_a} \). Measuring \( V_0, V \) and \( x \). Find the best value of \( V_0 \), which minimizes \( \sigma_{x_a} \).

8. The angle variation \( \theta \) of a wheel rotating under a constant acceleration, is measured at different times \( t \). A plot of \( \theta \) vs. \( t^2 \) of the data is fitted with a straight line \( y = a + bx \). Supposing that the initial angular velocity was zero, what are \( a \) and \( b \)? What is the value of \( \sigma_{t^2} \) for \( t = (1.32 \pm 0.02) \text{s} \)?

9. Plot the following set of data by hand on linear graph paper.
Supposing that the best fitting curve of experimental data is a straight line $y = ax + b$, graphically estimate the parameters $a$, and $b$.

Fit the data using an appropriate program (eg. matlab fit procedures), and analyze the differences plot. Try to reconcile any significant differences between the program fit parameters and your own.

10. Plot the following data by hand on both linear and semi-log graph paper:

<table>
<thead>
<tr>
<th>$x$ (a.u.)</th>
<th>16</th>
<th>44</th>
<th>76</th>
<th>87</th>
<th>110</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$ (a.u.)</td>
<td>0.037</td>
<td>0.097</td>
<td>0.25</td>
<td>0.41</td>
<td>0.80</td>
</tr>
</tbody>
</table>

(Uncertainties on the $y$ values are 6% of the value; i.e. $\sigma_y = 0.06y$.)

Determine a relationship between $x$ and $y$, and graphically estimate the uncertainties in all the fit parameters.

Plot and fit the data using an appropriate computer program (eg. matlab fit procedures), and compare with your previous results.
Chapter 2

The Maxwell Top

2.1 Introduction

In this chapter we want to study some particular cases of rigid body dynamics, which have a rotational symmetry around one axis, the so-called top, or Maxwell top.\(^1\)

In general, to solve the dynamics of a rigid body we must apply the second law of dynamics, i.e.

\[
\frac{d\vec{L}}{dt} = \vec{\tau},
\]

where \(\vec{\tau}\) is the external torque acting on the body, and \(\vec{L}\) is its angular momentum. For a solid body rotating around one of its axis of symmetry \(\hat{z}\) (more generally, around any of its three principal axes), with angular velocity \(\dot{\theta}\), \(\vec{L}\) is given by\(^2\)

\[
\vec{L} = I\dot{\theta}\hat{z},
\]

(2.1)

where \(I\) is the moment of inertia around the \(\hat{z}\) axis. \(I\) is

\[
I = \int (x^2 + y^2)\,dm,
\]

where \(x\) and \(y\) are the coordinates of the mass \(dm\). In this particular case,

\(^1\)The study of the top general equations of motion is quite complicated and is one of the main topics of a classical mechanics course.

\(^2\)The dot above the symbol stands for the derivative with respect to the time \(t\). The number of dots indicates the order of derivation.
the second law of dynamics assumes the simpler form

\[ I \ddot{\theta} = \tau. \]

\section{Some Relevant Examples}

In this section we will study three particular cases of the Maxwell top dynamics, which will be used in the laboratory procedures.

\subsection{Angular Acceleration under a Constant Torque}

Let's consider a top, whose axis of symmetry is vertical, and a force \( \vec{F} \) applied tangent to the top's surface in the horizontal plane containing the top's center of mass (see figure 2.1). If \( \vec{F} \) remains constant in modulus and direction in the reference frame rotating with the top, the second law of dynamics assumes a very simple form, i.e.\(^3\)

\[ I \ddot{\theta} = rF, \]

\(^3\)we are neglecting the energy dissipation mechanisms, which are always present in any physical system.
where $r$ is the arm lever distance. Integrating the previous equation we get
\[
\theta(t) = \theta_0 + \dot{\theta}_0 t + \frac{1}{2} \ddot{\theta}_0 t^2, \quad \ddot{\theta}_0 = \frac{rF}{I}.
\] (2.2)

where $\theta_0$ is the initial angle, $\dot{\theta}_0$ the initial angular velocity, and $\ddot{\theta}_0$ is the angular acceleration which is also constant.

### 2.2.2 Top Suspended with a Torsional Rod

![Figure 2.2: Top suspended to a torsional rod.](image)

By suspending the top with a torsional rod (see figure 2.2) we will have a restoring torque (the torsional version of the Hooke’s law) given by the linear equation
\[
\tau = -k\theta,
\]
where $\theta$ is the angle in the horizontal plane measured from the equilibrium position. From the second law of dynamics, we will have
\[
I\ddot{\theta} = -k\theta,
\]
which is the equation of an harmonic oscillator, whose general solution is
\[
\theta(t) = \theta_0 \cos(\omega_0 t + \phi_0), \quad \omega_0^2 = \frac{k}{I}.
\]
CHAPTER 2. THE MAXWELL TOP

The top will oscillate sinusoidally around the vertical axis with angular frequency $\omega_0$. The constant $\omega_0$ is said to be the angular resonant frequency of the torsional pendulum.

2.2.3 Precession of the Top

![Precession of the Top](image)

Figure 2.3: Precession of the top. In this sketch $\vec{\tau}$ is parallel to the $y$ axis and is pointing to the negative direction of the $y$ axis. The vector $\vec{h}_{CM}$ is in the plane $Oxz$.

Let’s suppose now that the top has its tip constrained on a horizontal plane and is rotating around its axis $\hat{u}$ at a constant angular velocity $\omega \hat{u}$ (see figure 2.3).

If the rotation axis makes an angle $\phi$ with $\hat{u}$ axis, the modulus of the torque $\vec{\tau}$, due to the gravity force $M \vec{g}$, is

$$\tau = |\vec{h}_{CM} \times M \vec{g}| = h_{CM} M g \sin \phi,$$

(2.3)

where $\vec{h}_{CM}$ is the vector pointing to the top center of mass. Because $\vec{g}$ is always vertical, $\vec{\tau}$ must always lie in the horizontal plane.

Because $\omega \hat{u}$ is parallel to $\vec{h}_{CM}$ at all times $\vec{L}$ is parallel to $\vec{h}_{CM}$ also. It follows that $\vec{L}$ is perpendicular to $\vec{\tau}$ at all times.
For the second law of dynamics and because \( \vec{\tau} \) is always in the horizontal plane, the variation \( d\vec{L} \) of the angular momentum must be always in the horizontal plane. This implies that projection of \( \vec{L} \) along the vertical axis is constant and \( \vec{L} \) can only rotate about the vertical axis. As a consequence the component of \( \vec{L} \) in the horizontal plane is constant in modulus but not in direction.

Considering that the projection of \( \vec{L} \) in the horizontal plane is \( L \sin \phi = \text{const.} \), the variation \( dL \) of \( \vec{L} \) must be (see figure 2.4)

\[
dL = L \sin \phi d\alpha,
\]
where \( d\alpha \) is the infinitesimal angular variation in the horizontal plane.

Using the second law of dynamics and the previous expression, we get

\[
L \sin \phi \frac{d\alpha}{dt} = \tau,
\]

The derivative is indeed the angular velocity \( \Omega \) of the top around the vertical axis. Combining the previous expression with the (2.3) we get

\[
h_{CM} M g = L \Omega.
\]

Substituting the (2.1) into the previous equation \( (\dot{\theta} = \omega) \) we finally get

\[
\Omega = \frac{M g}{I \omega} h_{CM}, \tag{2.4}
\]

which shows that \( \Omega \) does not depend on the angle \( \phi \). \( \Omega \) is said to be the precession angular frequency and when \( \Omega \neq 0 \), the top is said to precess around the vertical axis. It is worthwhile to notice that the angular momentum modulus \( |\vec{L}| \) is conserved.

Figure 2.4: Projection and variation of the angular momentum in the horizontal plane
2.3 Experimental setup.

The Maxwell top is shown schematically in Fig. 2.5. The top floats on an air cushion which creates a thin “air film” (less than 80 µm) and considerably reduces the frictional losses of energy. The two drive jets give the top a small torque, which can be changed acting on the adjustable exhaust valve. The sliding mass \(m\) changes the position of the top center of mass along the top axis. Fig. 2.5 also shows some details of the air circuit which sustains the top, and creates a thin air film for friction reduction between the base and the top.

Some other instruments needed for the two-week experiment are the following:

- a balance to measure various masses,
- a tachometer to measure the top angular velocity about its axis,
- a ring to increase the top moment of inertia,
- a torsional rod to suspend the top,
- a quasi-frictionless pulley, and a 2g weight to apply a constant torque to the top.

2.3.1 Care and Use of the Experimental Apparatus

The air bearing is a particularly delicate device because of the air film thickness. Any scratch or dirt on the air bearing surfaces can compromise the use of the experimental apparatus.

These are the precautions that need to be taken:

- **TURN THE AIR SUPPLY TO 26 PSI BEFORE ANY OPERATION.**
- **NEVER LET THE TOP SIT ON THE AIR BEARING BASE WITHOUT AIR FLOW.**
- **DO NOT SWITCH TOPS. EACH TOP WORKS PROPERLY WITH JUST ONE BASE.**
- **DO NOT LET ANY OBJECT FALL DOWN INTO THE AIR BEARING CUP.**
Figure 2.5: Maxwell Top schematic vertical cross section and top view cross section. $O$ is the top’s pivoting point and the sketched axis is oriented as indicated by the arrow. This implies that $h_0$ is negative and $h$ is positive.
• DO NOT USE CLEAR SCOTCH TAPE TO ATTACH WIRES TO TOP’S CYLINDER. USE SCOTCH MASKING TAPE PROVIDED BY THE LABORATORY.

• ALWAYS REMOVE THE SCOTCH TAPE FROM THE TOP’S CYLINDER ONCE FINISHED.

• REMEMBER TO CLOSE THE AIR SUPPLY OUTPUT ONCE FINISHED.

2.4 First Laboratory Week

The purpose of the lab is to apply the two methods of measuring the moment of inertia, based on the theory explained in the previous sections, and compare and analyze the results.

2.4.1 Indirect Measurement of the Moment of Inertia Applying a Constant Torque

The top’s moment of inertia $I$ can be indirectly measured if we apply a known constant torque $rF$ to it, and measure the revolution time of the top.

In fact, measuring the elapsed time for $1, 2, 3, \ldots, n$ revolutions, and fitting the data to the equation (2.2), we can obtain the value of the parameter $\dot{\theta}_0$ and indeed $I$.

2.4.2 Indirect Measurement of the Moment of Inertia Using a Torsional Pendulum

Another way to make an indirect measurement of a rigid body moment of inertia $I$ is measuring the period $T$ of a torsional pendulum, whose bob is the rigid body itself. By adding to the bob another rigid body with the known moment of inertia $I_0$ and re-measuring $T$, it allows to compute $I$ without knowing the characteristic of the torsional rod.

In fact, the angular frequencies of the rotation about the axis of symmetry for the two cases are

\[
\omega_1^2 = \frac{k}{I}, \quad \omega_2^2 = \frac{k}{I + I_0},
\]
which combined together, and considering that $\omega_i = \frac{2\pi}{T_i}$, result in

$$I = \frac{I_0}{\left(\frac{T_2}{T_1}\right)^2 - 1}$$  \hspace{1cm} (2.5)

The value of $I_0$ can be obtained indirectly by the definition of moment inertia.

### 2.4.3 Propaedeutic Problems

1. Derive how the moment of inertia $I_0$ for a ring of inner radius $r$, outer radius $R$, and mass $M$, is given by

$$I_0 = \frac{1}{2}M(R^2 + r^2)$$  \hspace{1cm} (2.6)

2. If the ring has a mass $M = (5.000 \pm 0.003)$ kg, an outer diameter $D = (200.0 \pm 0.5)$ mm, and an inner diameter $d = (180.0 \pm 0.5)$ mm, compute $I_0$, $\sigma_{I_0}$, and the relative error $\sigma_{I_0}/I_0$.

3. The measurement of the oscillation period of a torsional pendulum with a stopwatch, produces an error due to the experimenter’s reaction time. Assuming that this error is $\sigma = 0.05$ s, the pendulum period is $T = 2$ s, and only one measurement is performed, how many periods must be measured to get a relative error $\sigma_T/T$ of $\pm 2\%$, $\pm 1\%$, and $\pm 0.1\%$?

4. In determining the top’s moment of inertia $I$ with the torsional pendulum, it is found that the oscillation period is $T_1 = (1.260 \pm 0.003)$ s, and with the added ring with moment of inertia $I_0$, is $T_2 = (1.750 \pm 0.002)$ s. Find the uncertainty in the measurement of $I$. Use the values of $I_0$ and $\sigma_{I_0}$ given in problem 2.

5. Remember to close the air supply output once finished.

### 2.4.4 Procedure (Top’s Moment of Inertia Measurements)

Remember to follow the directives written in section 2.3.1 (Care and Use of the Experimental Apparatus) before starting the procedure.
1. Determine the top moment of inertia $I$ using the torsional rod as shown in the figure below.

Use the provided reflective sensor and the matlab command `MTOscPeriod` to measure the torsional period. For example, to measure the period by averaging the periods measured during 20 s, type `MTOscPeriod(20);`

Type `help MTOscPeriod` for a more complete command description and usage.

2. Determine the top’s moment of inertia fitting the angular displacement v.s. elapsed time, when the top is under a constant torque. To realize this condition, attach a string with a 2g weight to the top’s cylinder, and run the wire over the air pulley, as shown in the figure below.
Adjust the exhaust valve to change the air-jets flow until the top reaches a state closest as possible to the equilibrium. Remove the string from the top and measure the revolution periods. To keep the torque as constant as possible, never readjust the air pressure. To obtain a quasi-frictionless pulley, set the air flow of the pulley in such way the free pulley turns at very low speed. Try to keep the top as vertical as possible.

Use the provided reflective sensor and the matlab command MTCounterSave to measure the revolutions. For example, to count the top’s revolutions for 80 s and save the time and revolutions number into a file called Top3Revs.Trial01.txt, type

```
MTCounterSave('Top3Revs.Trial01.txt',80);
```

Type `help MTCounterSave` for a more complete command description and usage.

**Remember to remove ALL the scotch tape from the top’s rim once finished.**

3. Compare the two measured values of the moment of inertia $I$.

4. Compare the value of the angular acceleration $\tilde{\theta}_0$ obtained from the fit, with the value obtained from the definition of $\tilde{\theta}_0$ using the moment of inertia $I$ calculated in point 1.

### 2.5 Second Laboratory Week

The purpose of this lab is to verify that the precession angular velocity $\Omega$ is independent of the angle $\phi$ and to study the $\Omega$ as a function of the top’s center of mass.

If the center of the sliding mass $m$ is placed at a distance $h$ from the top’s pivot, and $h_0$ is the position of the top’s center of mass without $m$, the new center of mass will be located at (see Fig. 2.5)

$$h_{CM} = \frac{h_0M + hm}{M + m}$$  \hspace{1cm} (2.7)

It is important to notice that $h_0$ is negative because it is below the pivot $O$, which is the origin of the reference frame chosen to compute $h_{CM}$. With
the addition of the mass \( m \), equation (2.4) becomes

\[
\Omega = \frac{(M + m)gh_{CM}}{I\omega}, \tag{2.8}
\]

where we have neglected the small increase in the moment of inertia \( I \) due to the mass \( m \). Inserting equation (2.7) into the equation (2.8), and after some algebra, we obtain

\[
\Omega = \frac{Mg}{I\omega} h_0 + \frac{mg}{I\omega} h. \tag{2.9}
\]

which relates the precession angular velocity to the sliding mass position \( h \).

If we impose

\[
h = h^* = -h_0 \frac{M}{m}, \tag{2.10}
\]

the angular velocity \( \Omega \) of the precession goes to zero. If the mass \( m \) is placed at \( h^* \), \( h_{CM} \) is zero and the torque vanishes, and therefore the top does not precess.

It is important to notice that the spindle length is such that we can change the sign of \( h_{CM} \).

### 2.5.1 Propaedeutic Problems.

1. The position of the top center of mass \( h_0 \) without the sliding mass \( m \), is negative (below the pivot point). Provide a sketch depicting the direction of the angular velocity \( \omega \) and the direction of the top precession.

2. Calculate the period of precession \( T \) for a top spinning at 5Hz (5 revolutions per second, \( \omega = 2\pi \times 5 \text{rad/s} \)) if \( m = 0.2186 \text{kg}, \ I = 4.66 \cdot 10^{-2} \text{kg m}^2 \), and \( h = h^* + 0.01 \text{m.} \)

3. Given the sliding mass \( m = 0.2186 \text{kg} \) with its outside diameter \( D = 0.033 \text{m} \), and inside diameter \( d = 0.016 \text{m} \), calculate the sliding mass moment of inertia \( I_m \). Is the statement under equation (2.8) justified?

4. A linear fit to \( \Omega \) versus \( h \) gives \( \Omega = a + bh \). What are \( a \) and \( b \) in terms of \( m, g, I, \omega \) and \( h^* \)? Supposing that \( \omega \) is constant during
each measurement but different every time we change \( h \), how can we rearrange equation 2.9 to still use a straight line as fitting function?

5. The precession period \( T \) is measured with the sliding mass removed, and for a given value of the angular velocity \( \omega \). Write the equation that gives \( h_0 \) in terms of \( \omega, T, M, I \) and \( g \).

![Diagram of precession experiment](image)

6. Using two points, A and B, to align the line of sight (see figure above), a careful student determines that the uncertainty of measuring where the spindle passes the pointer at A is \( \Delta l = \pm 2 \text{mm} \). If the radius of the precession orbit is \( R = 50 \text{mm} \), and the period is \( T = 60 \text{s} \), what uncertainty \( \sigma_T \) does this produce in the period \( T \)? What fraction is \( \sigma_T \) of the total period \( T \)?

### 2.5.2 Procedure (Precession Period Measurement)

Remember to follow the directives written in section 2.3.1 (Care and Use of the Experimental Apparatus) before starting the procedure. Use the reflective sensor and the program Tachometer available from the computer desktop to measure the revolution frequency of the top.

Setting the revolution frequency of the top at around 5Hz make the following measurements:
1. Without the sliding mass, demonstrate that top precession period $T$ is independent from the angle $\phi$ for constant value of $\omega$.

2. Using the previous measurements of the precession period $T$, of the angular revolution frequency $\omega$, and of the moment of inertia $I$ calculate $h^*$ and its uncertainty. Place the sliding mass at $h^*$ and confirm that the top does not precess.

3. Experimentally study the equation of the precession angular velocity $\Omega$ as a function to the sliding mass position $h$. Be sure that you measure the length of the sliding mass.

4. Compare the new measurement of $I$ obtainable from step 3 with the two ones of the previous week.

5. Calculate the value of $h^*$ obtainable from step 4 and compare it with the previous measurement.

6. Remember to close the air supply output once finished.
Chapter 3

The Inverted Pendulum

3.1 Introduction

The purpose of this lab is to explore the dynamics of the harmonic mechanical oscillator. To make things a bit more interesting, we will model and study the motion of an inverted pendulum (IP), which is a special type of tunable mechanical oscillator. As we will see below, the IP contains two restoring forces, one positive and one negative. By adjusting the relative strengths of these two forces, we can change the oscillation frequency of the pendulum over a wide range.

As usual (see section 3.2), we will first make a mathematical model of the IP, and then you will characterize the system by measuring various parameters in the model. Finally, you will observe the motion of the pendulum and see if it agrees with the model to within experimental uncertainties.

The IP is a fairly simple mechanical device, so you should be able to analyze and characterize the system almost completely. At the same time, the inverted pendulum exhibits some interesting dynamics, and it demonstrates several important principles in physics. Waves and oscillators are everywhere in physics and engineering, and one of the best ways to understand oscillatory phenomenon is to carefully analyze a relatively simple system like the inverted pendulum.
3.2 Modeling the Inverted Pendulum (IP)

3.2.1 The Simple Harmonic Oscillator

We begin our discussion with the most basic harmonic oscillator—a mass connected to an ideal spring. We can write the restoring force $F = -kx$ in this case, where $k$ is the spring constant. Combining this with Newton’s law, $F = ma = m\ddot{x}$, gives $\ddot{x} = -(k/m)x$, or

$$\ddot{x} + \omega_0^2 x = 0$$

with

$$\omega_0^2 = k/m$$

The general solution to this equation is $x(t) = A_1 \cos(\omega_0 t) + A_2 \sin(\omega_0 t)$, where $A_1$ and $A_2$ are constants. (You can plug $x(t)$ in yourself to see that it solves the equation.) Once we specify the initial conditions $x(0)$ and $\dot{x}(0)$, we can then calculate the constants $A_1$ and $A_2$. Alternatively, we can write the general solution as $x(t) = A \cos(\omega_0 t + \varphi)$, where $A$ and $\varphi$ are constants.

The math is simpler if we use a complex function $\tilde{x}(t)$ in the equation, in which case the solution becomes $\tilde{x}(t) = \tilde{A} e^{i\omega_0 t}$, where now $\tilde{A}$ is a complex constant. (Again, see that this solves the equation.) To get the actual motion of the oscillator, we then can take the real part (or the imaginary part), so $x(t) = \text{Re}[\tilde{x}(t)]\footnote{If you have not yet covered why this works in your other courses, see the EndNote at the end of this chapter.}

You should be aware that physicists and engineers have become quite cavalier with this complex notation. We often write that the harmonic oscillator has the solution $x(t) = A e^{i\omega_0 t}$ without specifying what is complex and what is real. This is lazy shorthand, and it makes sense once you become more familiar with the dynamics of simple harmonic motion.

The Bottom Line: Equation 3.1 gives the equation of motion for a simple harmonic oscillator. The easiest way to solve this equation is using the complex notation, giving the solution $x(t) = A e^{i\omega_0 t}$.

3.2.2 The Simple Pendulum

The next step in our analysis is to look at a simple pendulum. Assume a mass $m$ at the end of a massless string of a string of length $\ell$. Gravity
exerts a force $mg$ downward on the mass. We can write this force as the vector sum of two forces: a force $mg \cos \theta$ parallel to the string and a force $mg \sin \theta$ perpendicular to the string, where $\theta$ is the pendulum angle. (You should draw a picture and see for yourself that this is correct.) The force along the string is exactly counteracted by the tension in the string, while the perpendicular force gives us the equation of motion

$$F_{\text{perp}} = -mg \sin \theta = m\ell \ddot{\theta} \quad \text{(3.2)}$$

so $\ddot{\theta} + (g/\ell) \sin \theta = 0$

As it stands, this equation has no simple analytic solution. However we can use $\sin \theta \approx \theta$ for small $\theta$, which gives the harmonic oscillator equation

$$\ddot{\theta} + \omega_0^2 \theta = 0 \quad \text{(3.3)}$$

where $\omega_0^2 = g/\ell \quad \text{(3.4)}$

The Bottom Line: A pendulum exhibits simple harmonic motion described by Equation 3.3, but only in the limit of small angles.

### 3.2.3 The Simple Inverted Pendulum

Our model for the inverted pendulum is shown in Figure 3.1. Assuming for the moment that the pendulum leg has zero mass, then gravity exerts a force

$$F_{\text{perp}} = +Mg \sin \theta \approx Mg\theta \quad \text{(3.5)}$$

where $F_{\text{perp}}$ is the component of the gravitational force perpendicular to the leg, and $M$ is the mass at the end of the leg. The force is positive, so gravity tends to make the inverted pendulum tip over, as you would expect.

In addition to gravity, we also have a flex joint at the bottom of the leg that is essentially a spring that tries to keep the pendulum upright. The force from this spring is given by Hooke’s law, which we can write

$$F_{\text{spring}} = -kx = -k\ell \theta \quad \text{(3.6)}$$
where $\ell$ is the length of the leg.

The equation of motion for the mass $M$ is then

$$M\ddot{x} = F_{\text{perp}} + F_{\text{spring}}$$

$$M\ell\ddot{\theta} = Mg\theta - k\ell\theta$$

and rearranging gives

$$\ddot{\theta} + \omega_0^2 \theta = 0$$

where $\omega_0^2 = \frac{k}{M} - \frac{g}{\ell}$

If $\omega_0^2$ is positive, then the inverted pendulum exhibits simple harmonic motion $\theta(t) = Ae^{i\omega_0 t}$. If $\omega_0^2$ is negative (for example, if the spring is too weak, or the top mass is too great), then the pendulum simply falls over.

The Bottom Line: A simple inverted pendulum (IP) exhibits simple harmonic motion described by Equation 3.8. The restoring force is supplied by a spring at the bottom of the IP, and there is also a negative restoring force from gravity. The resonant frequency can be tuned by changing the mass $M$ on top of the pendulum.
3.2. MODELING THE INVERTED PENDULUM (IP)

3.2.4 A Better Model of the Inverted Pendulum

The simple model above is unfortunately not good enough to describe the real inverted pendulum in the lab. We need to include a nonzero mass \( m \) for the leg. In this case it is best to start with Newton’s law in angular coordinates

\[ I_{\text{tot}} \ddot{\theta} = \tau_{\text{tot}} \]

(3.10)

where \( I \) is the total moment of inertia of the pendulum about the pivot point and \( \tau \) is the sum of all the relevant torques. The moment of inertia of the large mass is \( I_M = M\ell^2 \), while the moment of inertia of a thin rod pivoting about one end (you can look it up, or calculate it) is \( I_{\text{leg}} = m\ell^2 / 3 \). Thus

\[ I_{\text{tot}} = M\ell^2 + \frac{m\ell^2}{3} \]

(3.11)

\[ = \left( M + \frac{m}{3} \right) \ell^2 \]

The torque consists of three components

\[ \tau_{\text{tot}} = \tau_M + \tau_{\text{leg}} + \tau_{\text{spring}} \]

(3.12)

\[ = Mg\ell \sin \theta + mg \left( \frac{\ell}{2} \right) \sin \theta - k\ell^2 \theta \]

The first term comes from the usual expression for torque \( \tau = r \times F \), where \( F \) is the gravitational force on the mass \( M \), and \( r \) is the distance between the mass and the pivot point. The second term is similar, using \( r = \ell / 2 \) for the center-of-mass of the leg. The last term derives from \( F_{\text{spring}} = -k\ell \theta \) above, converted to give a torque about the pivot point.

Using \( \sin \theta \approx \theta \), this becomes

\[ \tau_{\text{tot}} \approx Mg\ell \theta + mg \left( \frac{\ell}{2} \right) \theta - k\ell^2 \theta \]

(3.13)

\[ \approx \left[ Mg\ell + \frac{mg\ell}{2} - k\ell^2 \right] \theta \]

and the equation of motion becomes

\[ I_{\text{tot}} \ddot{\theta} = \tau_{\text{tot}} \]

(3.14)

\[ \left( M + \frac{m}{3} \right) \ell^2 \ddot{\theta} = \left[ Mg\ell + \frac{mg\ell}{2} - k\ell^2 \right] \theta \]

(3.15)
which is a simple harmonic oscillator with

\[ \omega_0^2 = \frac{k\ell^2 - Mg\ell - \frac{mg\ell}{2}}{(M + \frac{m}{2})\ell^2} \]  

(3.16)

This expression gives us an oscillation frequency that better describes our real inverted pendulum. If we let \( m = 0 \), you can see that this becomes

\[ \omega_0^2(m = 0) = \frac{k}{M} - \frac{g}{\ell} \]  

(3.17)

which is the angular frequency of the simple inverted pendulum described in the previous section. If we remove the top mass entirely, so that \( M = 0 \), you can verify that

\[ \omega_0^2(M = 0) = \frac{3k}{m} - \frac{3g}{2\ell} \]  

(3.18)

The Bottom Line: The math gets a bit more complicated when the leg mass \( m \) is not negligible. The resonance frequency of the IP is then given by Equation 3.16. This reduces to Equation 3.17 when \( m = 0 \), and to Equation 3.18 when \( M = 0 \).

3.3 The Damped Harmonic Oscillator

To describe our real pendulum in the lab, we will have to include damping in the equation of motion. The major source of damping comes from the flex joint which is unable to give all the energy back during its bending motion. One way to account for this, is to use a complex spring constant given by

\[ \tilde{k} = k(1 + i\phi) \]  

(3.19)

where \( k \) is the normal (real) spring constant and \( \phi \) (also real) is called the loss angle. Looking at a simple harmonic oscillator, the equation of motion becomes

\[ m\ddot{x} = -k(1 + i\phi)x \]  

(3.20)

which we can write

\[ \ddot{x} + \omega_{\text{damped}}^2 x = 0 \]  

(3.21)

with \( \omega_{\text{damped}}^2 = \frac{k(1 + i\phi)}{m} \)  

(3.22)
3.4. THE DRIVEN HARMONIC OSCILLATOR

If the loss angle is small, \( \phi \ll 1 \), we can do a Taylor expansion to get the approximation

\[
\omega_{\text{damped}} = \sqrt{\frac{k}{m}} (1 + i\phi)^{1/2} \approx \sqrt{\frac{k}{m}} (1 + i\frac{\phi}{2}) = \omega_0 + i\alpha \]

with \( \alpha = \frac{\phi \omega_0}{2} \)

Putting all this together, the motion of a weakly damped harmonic oscillator becomes

\[
\ddot{x}(t) = \ddot{A} e^{-\alpha t} e^{i\omega_0 t} \]

which here \( \ddot{A} \) is a complex constant. If we take the real part, this becomes

\[
x(t) = A e^{-\alpha t} \cos(\omega_0 t + \phi) \]

This is the normal harmonic oscillator solution, but now we have the extra \( e^{-\alpha t} \) term that describes the exponential decay of the motion.

We often refer to the quality factor \( Q \) of an oscillator, which is defined as

\[
Q = \omega \frac{\text{Energy stored}}{\text{Power loss}}
\]

Note that \( Q \) is a dimensionless number. For our case this becomes (the derivation is left for the reader)

\[
Q \approx \frac{\omega_0}{2\alpha} = \frac{1}{\phi}
\]

The Bottom Line: We can model damping in a harmonic oscillator by introducing a complex spring constant. Solving the equation of motion then gives damped oscillations, given by Equations 3.27 and 3.28 when the damping is weak.

3.4 The Driven Harmonic Oscillator

If we drive a simple harmonic oscillator with an external oscillatory force \( F_0 e^{i\omega t} \), then the equation of motion becomes

\[
\ddot{x} + \omega_{\text{damped}}^2 x = \frac{F_0}{m} e^{i\omega t}
\]
where $\omega$ is the angular frequency of the drive force and $F_0$ is the force amplitude. (As above, $\omega_{\text{damped}} = \omega_0 + i \alpha$.) Analyzing this shows that the system first exhibits a transient behavior that lasts a time of order

$$t_{\text{transient}} \approx \alpha^{-1} \approx 2Q/\omega_0 \quad (3.32)$$

During this time, the motion is quite complicated and depends on the initial conditions and the phase of the applied force.

Because of the damping, the transient behavior eventually dies away, however, and for \( t \gg t_{\text{transient}} \) the system settles into a steady-state behavior, where the motion is given by

$$x(t) = X e^{i\omega t} \quad (3.33)$$

In other words, in steady-state the system oscillates with the same frequency as the applied force, regardless of the natural angular frequency $\omega_0$. Plugging this $x(t)$ into the equation of motion quickly gives us

$$X = \frac{F_0/m}{\omega_{\text{damped}}^2 - \omega^2} \quad (3.34)$$

Since $X$ is a complex constant, it gives the amplitude $|X|$ and phase $\arg(X)$ of the motion. When the damping is small ($\alpha \ll \omega_0$), we can write $\omega_{\text{damped}}^2 \approx \omega_0^2 + 2i\alpha\omega_0$, giving

$$X \approx \frac{F_0/m}{(\omega_0^2 - \omega^2) + 2i\alpha\omega_0} \quad (3.35)$$

and the amplitude of the driven oscillations becomes

$$|X(\omega)| = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\alpha^2\omega_0^4}} \quad (3.36)$$

Note that the driven oscillations have the highest amplitude on resonance ($\omega = \omega_0$), and the peak amplitude is highest when the damping is lowest.

The Bottom Line: Once the transient motions have died away, a harmonically driven oscillator settles into a steady-state motion exhibiting oscillation at the same frequency as the drive. The amplitude is highest on resonance (when $\omega = \omega_0$) and when the damping is weak, as given by Equation 3.36.
3.5 The Transfer Function

For part of this lab you will shake the base of the inverted pendulum and observe the response. To examine this theoretically, we can look first at the simpler case of a normal pendulum in the small-angle approximation (when doing theory, always start with the simplest case and work up). The force on the pendulum bob (see Equation 3.2) can be written

\[ F = -mgx/\ell \]  

(3.37)

where \( x \) is the horizontal position of the pendulum and \( \ell \) is the length. If we shake the top support of the pendulum with a sinusoidal motion,

\[ x_{\text{top}} = X_{\text{drive}}e^{i\omega t} \],

then this becomes

\[ F = -mg \left( x - x_{\text{top}} \right)/\ell \]  

(3.38)

\[ \ddot{x} + \omega_0^2 x = \omega_0^2 x_{\text{top}} \]  

(3.39)

where \( \omega_0^2 = g/\ell \). With damping this becomes

\[ \ddot{x} + \omega_{\text{damped}}^2 x = \omega_0^2 x_{\text{top}} \]  

(3.40)

\[ = \omega_0^2 X_{\text{drive}}e^{i\omega t} \]  

(3.41)

which is essentially the same as Equation 3.31 for a driven harmonic oscillator. From the discussion above, we know that this equation has a steady-state solution with \( x = Xe^{i\omega t} \). It is customary to define the transfer function

\[ H(\omega) = \frac{X}{X_{\text{drive}}} \]  

(3.42)

which in this case is the ratio of the motion of the pendulum bob to the motion of the top support. Since \( H \) is complex, it gives the ratio of the amplitudes of the motions and their relative phase.

For the simple pendulum case, Equation 3.35 gives us (verify this for yourself)

\[ H(\omega) \approx \frac{\omega_0^2}{(\omega_0^2 - \omega^2) + 2i\alpha \omega} \]  

(3.43)

At low frequencies \( (\omega \ll \omega_0) \) and small damping \( (\alpha \ll \omega_0) \), this becomes \( H \approx 1 \), as you would expect (to see this, consider a mass on a string,
and shake the string as you hold it in your hand). At high frequencies \((\omega \gg \omega_0)\), this becomes \(H(\omega) \approx -\omega_0^2 / \omega^2\), so the motion of the bob is 180 degrees out of phase with the motion of the top support (try it).

The Bottom Line: The transfer function gives the complex ratio of two motions, and it is often used to characterize the behavior of a driven oscillator. Equation 3.43 shows one example for a simple pendulum. The motion of the inverted pendulum is a bit more interesting, as you will see when you measure \(H(\omega)\) in the lab.

### 3.6 The Inverted Pendulum Test Bench

#### 3.6.1 Care and Use of the Apparatus

The Inverted Pendulum hardware is not indestructible, so please treat it with respect. Ask your instructor if you think something is broken or otherwise amiss. The flex joint is particularly delicate, and bending it to large angles can cause irreparable damage. Follow these precautions:

1. **Never let the IP oscillate without the travel limiter.**
2. **Do not let the IP leg fall.**
3. **Do not disassemble the IP without assistance from your instructor.**

### 3.7 The Lab - First Week

#### 3.7.1 Pre-Lab Problems

1. Rewrite Equation 3.16 using the “rotational stiffness” variable \(\kappa = k\ell^2\) for the spring constant. What are the units of \(k\), \(\kappa\)?
2. Determine the length of an IP leg with a load of \(M = 0.383\) kg, flex joint rotational stiffness \(\kappa = 2.5\) Nm/rad, oscillation period \(T = 10\) seconds, and negligible leg mass, \(m = 0\). Compute the length of a simple pendulum with the same oscillation frequency. Assume \(g = 9.81\) m/s\(^2\).
3. For the IP in Problem 2, calculate the mass difference \(\Delta M\) needed to change the period from 10 seconds to 100 seconds. At what mass \(M^*\) does the period go to infinity?
4. For the IP in Problem 2, if the loss angle is $\phi = 10^{-2}$, how long does it take for the motion to damp down to 1% of its starting amplitude?

### 3.7.2 In-Lab Exercises

#### 3.7.2.1 Getting Started

Please read down to the end of this section before beginning your work. Once you begin in the lab, record all your data and other notes in your notebook as you proceed. Print out relevant graphs and tape them into your notebook as well.
CHAPTER 3. THE INVERTED PENDULUM

Step 1. All the measurements this week are done with the actuator platform locked in place. Use the attached thumb screws to lock the platform (see your instructor if you are not sure about this). If the platform is securely locked, it should not rattle if you shake it gently.

![Figure 3.3: IP leg with the flex joint and the load device.](image)

3.7.2.2 The IP Leg

When adding mass to the top of the leg, add weights symmetrically on the load device (see Figure 3.3). This ensures that the center-of-mass of the added weight is always positioned at the same distance from the pivot point (i.e., this ensures that $\ell$ stays constant as you change $M$).

Step 2. Using the spare leg in the lab, measure the length $\ell$ and mass $m$ of the leg, including measurement uncertainties (error bars). Measure $\ell$ from the center of the flex joint to the center-of-mass of the added weight. Note that you cannot actually measure $\ell$ very well because of the large size of the flex joint. Use common sense to estimate an uncertainty in $\ell$, based on the length of the flex joint and how accurately the load mass can be placed. Note that your measurement of $m$ does not include the mass of the magnet assembly on the IP, which you can take to be $M_{\text{magnet}} = 6$ grams.

3.7.2.3 Resonant Frequency versus Load

Step 3. With no mass on the top of the leg ($M = 0$), measure the oscillation period $P = 1/2\pi\omega_1$ (with an error bar) of the IP using the Matlab data-acquisition function IPRingDown. Use Equation 3.16 to estimate $k$ based
on your frequency measurement. Use your direct measurements of \( m \) and \( \ell \). Assuming that the theory in Equation 3.16 is exact, use standard error propagation methods to estimate an error bar for \( k \) from the errors in the other quantities.

Step 4. Make additional measurements of \( P \) for different values of \( M \) starting with the largest mass you will plan to use. Add washers symmetrically about the center-of-mass (see Figure 3.3). Note that you may have to level the IP as you add more mass using the three leveling thumb screws; ask your instructor if you need help. As the top mass \( M \) increases, the IP is more likely to tip over, so the leveling of the apparatus is quite important.

For each measurement of \( P \), measure \( M \) using the scale in the lab. Take at least 5-7 data points with error bars. Note that the errors in the measurement of \( P \) are larger for larger \( P \).

Step 5. Use Equation 3.16 to compute a theoretical curve to add to your data plot. One way to accomplish this is by creating a short Matlab program (a "m file" or script file). Here is an example (ask your instructor if you need help on how to edit and save your own program in Matlab):
% Dec 2, 2012
% IP_PeriodVersusMass.m
% Sample program to plot IP data with theory
% replace ? with values%
clear; %clear all variables

%% Measurements
m = ?; % kg, measured shaft mass
l = ?; % m, measured shaft length
M_Magnet = 0.010; % kg, extra mass from magnet assembly
M_measured = [? ? ? ? ?] + M_Magnet; % kg, measured added mass
T_measured = [? ? ? ? ?]; % s, measured period

%% Constants
g = 9.81; % m/s^2, gravitational constant

%% Guess for k
k = input('Enter educated guess for k: ');

%% Compute the theoretical curve
M = 0.01:.001:.35; %kg, mass vector for theoretical curve
w0 = sqrt((k*l^2 - M*g*l - m*g*l/2) ./ ((M + m/3)*l^2)); % rad/s
T = 2*pi./w0; %s, theoretical oscillation period

close all; % close all figures

%% Plot Theoretical data in semilog scale
semilogy(M,T,'b')
hold('on') %hold the current so new plot can be added

%% Plot experimental data
plot(M_measured,T_measured,'o')
xlabel('Added mass M, [kg]')
ylabel('IP period T, [s]')
axis('tight')
grid('on')
hold('off')

Try putting in different values of k to see how the theory changes. You should get a nice fit using your calculated value of k, although a different value may fit better. The real IP is not exactly the same as depicted in
theory, so there will be some systematic discrepancies, and some parameter estimation (IP length for example) can have systematic errors. Plot the data with theory for your best subjective estimate of $k$, and add the plot to your notebook.

Step 6. Create a difference plot of your experimental data points and the theoretical curve to re-estimate $k$ (hint: copy and modify the Matlab program you already wrote and use the PlotBoxesLinLin command to plot your experimental data points.)

Step 7. From observing how well the theory fits the data for different guesses for $k$, estimate your best-fit $k$ along with an error bar (hint: use the difference plot). How does this compare with your estimate of $k$ in Step 6 and 3?

3.8 The Lab - Second Week

3.8.1 Pre-Lab Problems

1. Using matlab or your preferred tool, plot the amplitude response of a driven harmonic oscillator, given by Equation 3.36. Assume a resonance frequency $\omega_0 = 2\pi$ Hz, $F_0/m = 1$, and plot the amplitude as a function of angular frequency $\omega$. Make three plots (preferably all on a single graph) using $Q = 1, 10, \text{ and } 100$. Plot all three on a linear-linear plot, then plot all three again on a log-log plot. Note that the response shows a power-law behavior ($x \sim \omega^{-2}$) at high frequencies, and this appears as a straight line in the log-log plots. Consult the matlab plot script examples posted on the ph3 web pages.

2. Let $\Delta \omega$ be the FWHM (full-width at half-maximum) width of $|A(\omega)|^2$, i.e. when $\omega = \omega_0 \pm \Delta \omega/2$, then $|A(\omega_0 \pm \Delta \omega/2)|^2 = |A(\omega_0)|^2 / 2$. In the limit of high $Q$, find $Q$ as function of $\omega_0$ and $\Delta \omega$. (Hint: use Equation 3.36 and find an approximate formula for $|A(\omega)|^2$ by imposing $\omega = \omega_0 + \epsilon$, and neglecting terms of the order $\epsilon^3$ and higher .

3. In the lab we will measure $\dot{X}(v)$ and $\dot{X}_{\text{drive}}(v)$, the velocities of the pendulum bob and support platform. Show that $H(v) = X(v)/X_{\text{drive}}(v) = \dot{X}(v)/\dot{X}_{\text{drive}}(v)$. 
3.8.2 In-Lab Exercises

3.8.2.1 Inverted Pendulum Loss Angle

Step 1. With the actuator platform locked (same as last week), use IPRing-Down to observe the motion of the inverted pendulum with no added mass and no added damping. Put a plot of the IP motion $x_{IP}(t)$ in your notebook, and estimate the loss angle $\phi$ from the time it takes the motion to damp away. Now add the aluminum damper plate to the top of the IP assembly (ask your instructor). As the pendulum swings, the damping magnet induces a current in the aluminum plate, which heats the plate and extracts energy from the IP motion. With the aluminum plate in place (about 6 mm from the magnet), again plot $x_{IP}(t)$ and estimate $\phi$ from a fit. Unscrew the magnet holder tube a bit so the magnet is quite close to the plate, and again plot $x_{IP}(t)$ and estimate $\phi$. For each case, how long is $t_{\text{transient}}$ – the time needed for a driven system to reach steady-state motion (see Equation 3.32)?

3.8.2.2 The Transfer Function

The rest of the lab is done with the actuator platform unlocked; ask your instructor if you need help with this. Drive the motion of the platform using a sinusoidal voltage from the signal generator. Be sure to set the "sweep" button to EXT (which turns off the sweep feature of the signal generator). With a high signal amplitude you should be able to see the platform oscillating, and you can see the frequency change as you change the frequency of the drive voltage. MAKE SURE THAT THE MOTION OF THE PLATFORM IS SINUSOIDAL.

With no added mass on the IP, use the Matlab program IPTransmissibility to measure the motion of the platform $x_{\text{platform}}(t)$ and the motion of the top mass $x_{IP}(t)$. Use the ‘t’ command to view both motions as a function of time. If both do not show simple sinusoidal oscillations, turn down the drive amplitude. The program IPTransmissibility will collect data for you as you change the drive frequency, and it will compare $x_{IP}(t)$ and $x_{\text{platform}}(t)$ to give you the amplitude and phase of the transfer function as a function of frequency.

Step 2. Collect data at a sufficient number of drive frequencies to map out the transfer function $H(\nu)$ as a function of frequency. Put a plot of $H(\nu)$ in your notebook, and qualitatively explain the features of $H(\nu)$.
3.9. **EndNote: Using Complex Functions to Solve Real Equations**

(both amplitude and phase), given the above discussion of a driven harmonic oscillator.

**Step 3.** Now add mass to the IP so the resonant frequency is between 0.5 and 1 Hz (use your data from last week to see what $M$ is needed). Place the damper plate about 3 mm from the damping magnet. Starting at low frequencies, use IPTransmissibility to again map out $H(\nu)$. Again, use the \textquoteleft t\textquoteright~command to make sure the motions are sinusoidal. Above the resonance frequency of the IP, you will need to turn up the drive amplitude so the signal-to-noise in the measurements remains adequate. Be sure to continue your measurements to at least 10 Hz. Put a plot of $H(\nu)$ in your notebook, and again qualitatively explain the features of $H(\nu)$ (both amplitude and phase). [Hint: an understanding of the \textquotedblleft center of percussion\textquotedblright~of a bar will be useful for explaining the high-frequency behavior.]

### 3.9 EndNote: Using Complex Functions to Solve Real Equations

Physicists and engineers often use complex functions to solve real equations, with the understanding that you take the real part at the end. Why does this work? And why do we even do this? We can demonstrate with the simple harmonic oscillator. Start with the equation of motion $\ddot{x} + \omega_0^2 x = 0$, and let us solve this using a complex function: $x = \alpha + i\beta$, where $\alpha(t)$ and $\beta(t)$ are real functions. If you plug this in, you will see that $\ddot{x} + \omega_0^2 x = 0$ becomes

\[
\begin{align*}
(\ddot{\alpha} + i\ddot{\beta}) + \omega_0^2 (\alpha + i\beta) &= 0 \\
(\ddot{\alpha} + \omega_0^2 \alpha) + i (\ddot{\beta} + \omega_0^2 \beta) &= 0
\end{align*}
\]

Since a complex number equals zero only if both the real and imaginary parts equal zero, we see that $\ddot{x} + \omega_0^2 x = 0$ implies that both $\ddot{\alpha} + \omega_0^2 \alpha = 0$ and $\ddot{\beta} + \omega_0^2 \beta = 0$. In other words, both the real and imaginary parts of $x(t)$ satisfy the original equation.

So we have a procedure: try using a complex function to solve the original equation. If this works, then taking the real part of the solution gives a real function that also solves the same differential equation. (If in doubt, then verify directly that the real part solves the equation.)
CHAPTER 3. THE INVERTED PENDULUM

Why do we go to the trouble of using complex functions to solve a real equation? Because differential equations are often easier to solve when we assume complex functions (it seems counter-intuitive, but it’s true). The function $e^{i\omega_0 t}$ is a simple exponential, and the derivative of an exponential is another exponential – that makes things simple. In contrast, cosines and sines are more difficult to work with.

In the case of the simple harmonic oscillator, the solution $x(t) = Ae^{i\omega t}$ has a natural interpretation. The length and angle of the $A$ vector (in the complex plane) give the amplitude and phase of the oscillations.

You should note, however, that this only works for linear equations. If our equation were $\ddot{x} + \omega_0^2x + \gamma x^2 = 0$, for example, then using complex functions would not have the same benefits. In fact there is no simple solution to this equation, complex or otherwise. This equation describes a nonlinear oscillator, and nonlinear oscillators exhibit a fascinating dynamics with interesting behaviors that people still study to this day.
Chapter 4

Direct Current Network Theory

4.1 Electronic Networks

An electronic network or circuit is a set of electronic components/devices connected together to modify and transmit/transfer energy/information. This information is generally called electric signal or simply signal. To graphically represent a network, we use a set of coded symbols with terminals for devices and lines for connections. These lines propagate the signal among the devices without changing it. Devices change the propagation of the electric signals instead.

Quantities defining this propagation are the voltages $V$ across the devices and currents $I$ flowing through them.

Solving an electronic network means determine the currents or the voltages on each point of it.

To make the understanding of network basic theorems easier, some more or less intuitive definitions must be stated.

4.1.1 Network Definitions

A network node is a point where more than two network lines connect.

A network loop, or mesh, is any closed network line. To determine a mesh it is sufficient to start from any point of the circuit and come back running through the network to the same point without passing through a same point.

Figure 4.1 shows a generic portion of a network with 4 visible nodes,
4.1.2 Series and Parallel

Let's consider the two different connection topologies shown in figure 4.2, the parallel and the series connections.

A set of components is said to be in series if the current flowing through them and anywhere in the circuit is the same.

A set of components is said to be in parallel if the voltage difference between them is the same.

Figure 4.2: Considering the points A, and B, the components of the left circuit are in parallel, and those ones in the right circuit are in series.
4.1.3 Active and Passive Components

Circuit components can be divided into two categories: active and passive components. Active components are those devices that feed energy into the network. Voltage and current sources are active components. Amplifiers are also considered active components.

Passive components are those components that do not feed energy to the network. Resistors, capacitors, inductors are typical passive components.

In general, both active and passive components dissipate energy.

4.2 Kirchhoff’s Laws

In this section the two Kirchhoff’s laws, which are fundamental for the solution of an electronic circuit are stated here.

**Kirchhoff’s Voltage Law (KVL):** The algebraic sum of the voltage difference $v_k$ around a mesh must be equal to zero at all times, i.e.

$$\sum_k v(t) = 0$$

**Kirchhoff’s Current Law (KCL):** The algebraic sum of the currents $i_k$ entering and leaving a node must be equal to zero at all times, i.e.

$$\sum_k i_k(t) = 0.$$  

4.3 Resistors (Ohm’s Law)

It is experimentally known that if we apply a voltage $V$ across a metal (a conductor), we will measure an electric current $I$ flowing through it. $V$
results to be proportional to \( I \) by a constant \( R \), named electric resistance of the conductor. In other terms we have

\[
\frac{V}{I} = R, \tag{4.1}
\]

which is called Ohm’s law. The unit for the resistance is the “Ohm” whose symbol is the Greek letter \( \Omega \). It follows from Ohm’s law that \( [\Omega] = [V/A] = \text{"Volt per Ampere"} \).

In a metallic conductor, \( R \) is proportional to the conductor length \( l \) and inversely proportional to the cross-section \( s \), i.e.

\[
R = \rho \frac{l}{s}.
\]

\( \rho \) is the conductor resistivity and it depends on the metal and its impurities. A device which follows Ohm’s Law is said to be a resistor and its symbol is shown in figure 4.3.

The power \( P \) dissipated by the resistor with resistance \( R \) is

\[
P = VI, \quad \Rightarrow \quad P = \frac{V^2}{R} = RI^2.
\]

Let’s define some properties of the resistance of conductors.

### 4.3.1 Resistors in Series

![Figure 4.4: Resistor in series](image)

The resistance \( R_{\text{tot}} \) of a set of resistor \( R_1, R_2, \ldots, R_n \), connected in series (see figure 4.4), is equal to the sum of the resistances.
4.3. RESISTORS (OHM’S LAW)

\[ R_{\text{tot}} = \sum_{k=1}^{n} R_k. \]

The previous formula is easy to demonstrate. Connecting the resistor series to a voltage source \( V \), and applying the (4.1) to each resistors we have

\[ V_1 = R_1 I, \quad V_2 = R_2 I, \ldots \quad V_n = R_n I, \]

where \( I \) is the current flowing through each resistor. Because of the KVL, the voltage difference \( V \) must be the sum of all the voltages differences, i.e.

\[ V = \sum_{k=1}^{n} R_k I = \left( \sum_{k=1}^{n} R_k \right) I. \]

4.3.2 Resistors in Parallel

The inverse of the resistance \( R_{\text{tot}} \) of a set of resistor \( R_1, R_2, \ldots, R_n \), connected in parallel (see figure 4.5), is equal to the sum of the inverse of the resistances

\[ \frac{1}{R_{\text{tot}}} = \sum_{k=1}^{n} \frac{1}{R_k}. \]

This law can be easily derived using (4.1) and the definition of components in parallel.

![Figure 4.5: Resistors in parallel.](image-url)
4.4 Capacitors

Let’s study another typical component of an electronic circuit, the capacitor. A capacitor is a system of two conductors which goes under full induction when a voltage difference is applied to the conductors. Each conductor will be charged with the same amount of charge \( Q \) with opposite sign. The ratio between the voltage difference \( V \) between the two conductors and the charge \( Q \)

\[
C = \frac{Q}{V}
\]

is constant and is said to be the capacitance \( C \) of the capacitor. The capacitance depends on the geometry of the conductors and on the interposed dielectric. The units for the capacitance is the “Faraday” whose symbol is the letter F. It follows that \([F] = [C/V] = \text{Coulomb per Volt}\).

4.4.1 Capacitors in Parallel

A parallel of capacitors \( C_1, C_2, \ldots, C_n \) is a capacitor whose capacitance \( C_{tot} \) is the sum of all the capacitances, i.e.

\[
C_{tot} = \sum_{k=1}^{n} C_k.
\]

The previous formula is easy to demonstrate. In fact, the total induced charge \( Q_{tot} \) on the capacitors side at the same potential is equal to the sum of all charges of those sides, i.e.

\[
Q_{tot} = \sum_{k=1}^{n} Q_k.
\]

Considering that the voltage difference \( V \) across each capacitor must be same, and dividing by \( V \) we obtain the (4.3).
4.4. CAPACITORS

4.4.2 Capacitors in Series

A series of capacitors $C_1, C_2, \ldots, C_n$ is a capacitor with capacitance $C_{tot}$ satisfying the following equation

$$\frac{1}{C_{tot}} = \sum_{k=1}^{n} \frac{1}{C_k}.$$ 

The demonstration of the previous equation is left as exercise (hint: in this case the induced charges are equal and not the voltage differences across the capacitors).
4.5 Ideal and Real Sources

4.5.1 Ideal Voltage Source

An ideal voltage source is a source able to deliver a given voltage difference \( V_s \) between its leads independently of the load \( R \) attached to it (see figure 4.9). It follows from Ohm’s law that a voltage source is able to produce any current \( I \) to keep constant the voltage difference \( V_s \) across the load \( R \). The symbol for the ideal voltage source is shown in figure 4.9.

Quite often, a real voltage source exhibits a linear dependency on the resistive load \( R \). It can be represented using an ideal voltage source \( V_s \) in series with a resistor \( R_s \) called input resistance of the source. Applying Ohm’s law, it can be easily shown that the voltage and current through the load \( R \) are

\[
V = \frac{R}{R + R_s} V_s, \quad I = \frac{V_s}{R + R_s}.
\]

If we assume

\[ R \gg R_s, \quad \Rightarrow \quad V \simeq V_s, \quad I \simeq \frac{V_s}{R}. \]

Under the previous condition, the real voltage source approximates the ideal case.

![Figure 4.9: Ideal voltage source.](image)

4.5.2 Ideal Current Source

An ideal current source is a source able to deliver a given current \( I_s \) which does not depend on the load \( R \) attached to it (see figure 4.10). It follows from Ohm’s law that an ideal current source is able to produce any voltage...
4.6. THE SEMICONDUCTOR JUNCTION (DIODE)

The difference $V$ across the load $R$ to keep $I_s$ constant. The symbol for the ideal current source is shown in figure 4.10.

A real current source exhibits a dependency on the resistive load $R$, which can be represented using an ideal current source $I_s$ in parallel with a resistor $R_s$. Applying Ohm’s law and the KCL, it can be easily shown that the voltage and current through the load $R$ are

$$I = \frac{R_s}{R + R_s}I_s, \quad V = \frac{R_s R}{R_s + R}I_s.$$

If we suppose $R_s \gg R_s \Rightarrow I \simeq I_s, \quad V \simeq R I_s \gg 0$

Under the previous condition, the real current source approximates the ideal case.

![Ideal current source diagram]

**Figure 4.10: Ideal current source.**

4.6 The Semiconductor Junction (Diode)

The *semiconductor junction* or *semiconductor diode* is a device which presents a non-linear behavior due to its peculiar conduction mechanism substantially different from the conduction in a metal.

If $I_D$ and $V_D$ are the current and the voltage difference across the junction, we will have

$$I_D(V_D) = I_0(e^{\frac{-qV_D}{k_B T}} - 1), \quad (4.4)$$

where $k_B = 1.3807 \cdot 10^{-23} \text{J/K}$ is the Boltzmann constant, $T$ the absolute temperature, $q = -1.60219 \cdot 10^{-19} \text{C}$ the electron charge, and $\eta$ a dimensionless parameter, which depends on the diode type. $I_0$ is the reverse saturation current.
Instead of following Ohm’s law, the semiconductor junction follows an exponential curve (the diode I-V Characteristic). Deviations from this law are negligible depending on the current magnitude and the diode characteristics.

Figure 4.11 shows the standard symbol for a semiconductor diode and the I-V characteristic. The break-down voltage $V_b$ reported in the same figure is the reverse voltage which essentially shorts circuit the junction. This behavior is not accounted by the equation (4.4).

A simplified model of the junction diode is that one of a perfect switch, i.e.

$$I_D(V) = \begin{cases} \infty & V \geq V_{on} \\ 0 & V < V_{on} \end{cases},$$

where $V_{on}$ is the diode turn-on voltage, which depends on the junction type and on the current magnitude. For current up to $I_D \sim 100\, \text{mA}$, silicon diodes have $V_{on} \simeq 0.6\, \text{V}$, and germanium diodes have $V_{on} \simeq 0.3\, \text{V}$.

For voltages greater than $V_{on}$, the diode is a short circuit (current is not limited by the diode) and is said to be forward biased. For smaller values it is an open circuit (current across the diode is zero) and is reverse biased.
4.7 Equivalent Networks

Quite often, the analysis of a network becomes easier by replacing part of it with an equivalent and simpler network or dividing it into simpler sub-networks. For example, the voltage divider is an equation easy to remember that allows to divide a complex circuit in two parts simplifying the search of the solution. Thévenin and Norton theorems, give us two methods to calculate equivalent simple circuits, which behave like the original circuit seen from two points of it. These techniques briefly explained in this section, will be used also in some of the next experiments.

4.7.1 Voltage Divider

The voltage divider equation is applicable every time we have a circuit which can be re-conducted in a series of two simple or complex components. The simplest case is the one shown in figure 4.12. Applying Ohm’s law, we have

\[
V_{Tot} = (R_1 + R_2)I \\
V_2 = R_2I
\]

and indeed

\[
V_2 = \frac{R_2}{R_1 + R_2}V_{Tot}
\]

which is the equation of the voltage divider. The equation still holds if we replace the resistance seen from the points A, B for \( R_1 \) and B, C for \( R_2 \).
4.7.2 Thévenin Theorem

Thévenin theorem allows to find an equivalent circuit of a network seen from two points A and B using the series of an ideal voltage source of voltage $V_{Th}$ and a resistor with resistance $R_{Th}$.

![Thévenin equivalent circuit illustration.](image)

The equivalence means that if we place a load $R_L$ between A and B in the original circuit (see figure 4.13) and measure the voltage $V_L$ and the current $I_L$ across the load, we will obtain exactly the same $V_L$ and $I_L$ if $R_L$ is placed in the equivalent circuit. This must be true for any load we connect to the points A and B.

The previous statement and the linearity of the circuit can be used to find $V_{Th}$ and $R_{Th}$. In fact, if we consider $R_L = \infty$ (open circuit, OC), we will have

$$V_{Th} = V_{OC}.$$ 

The Voltage $V_{Th}$ is just the voltage difference between the two leads A and B.

For $R_L = 0$ (short circuit, SC) we must have

$$I_{SC} = \frac{V_{Th}}{R_{Th}} = \frac{V_{OC}}{R_{Th}}.$$ 

and therefore

$$R_{Th} = \frac{V_{OC}}{I_{SC}}.$$ 

The last expression says that the Thévenin resistance is the resistance seen from the points A and B of the original circuit.

If the circuit is known the Thévenin parameter can be calculated in the case of the terminals A and B open. In fact, $V_{Th}$ is just the voltage
4.7. EQUIVALENT NETWORKS

across A and B of a known circuit. Replacing the ideal voltage sources with short circuits (their resistance is zero) and ideal current sources with open circuits (their resistance is infinite) we can calculate the resistance $R_{Th}$ seen from terminals A and B.

**Example:**

We want to find the Thévenin circuit of the network enclosed into the gray rectangle of figure 4.14. To find $R_{Th}$ and $V_{Th}$ we have to disconnect the circuit in the points A and B. In this case voltage difference between these two points, thanks to the voltage divider equation, is

$$V_{Th} = \frac{R_1}{R_1 + R_2} V_s.$$

Short circuiting $V_s$ we will have $R_1$ in parallel with $R_2$. The Thévenin resistance $R_{Th}$ will be indeed

$$R_{Th} = \frac{R_1 R_2}{R_1 + R_2}.$$

Considering the previous results, we can finally state Thévenin theorem as follows:

*Any circuit seen from two points can be replaced by a series of an ideal voltage source of voltage $V_{Th}$ and a resistor of resistance $R_{Th}$. $V_{Th}$ is the voltage difference*
between the two points of the original circuit. $R_{Th}$ is the resistance seen from these two points, short-circuiting all the ideal voltage generators and open-circuiting all the ideal current generators.

![Norton equivalent circuit illustration.](image)

**Figure 4.15: Norton equivalent circuit illustration.**

### 4.7.3 Norton Theorem

Any kind of active network seen from two points $A$ and $B$ can be replaced by an ideal current generator $I_{No}$ in parallel with a resistance $R_{No}$. The current $I_{No}$ corresponds to the short-circuit current of the two points $A$ and $B$. The Resistance $R_{No}$ is the Thévenin resistance $R_{No} = R_{Th}$.

The proof of this theorem is left as an exercise.

### 4.8 Resistor Color Code

Nominal values of resistances are coded using color bands around the resistors (see figure below). The bands identify digits and the exponent in base ten for the resistance value and the tolerance as explained in the following table:

<table>
<thead>
<tr>
<th>Band Number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4 (Tolerance band)</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 Bands</td>
<td>Digit</td>
<td>Digit</td>
<td>Exponent</td>
<td>Always 20%</td>
<td></td>
</tr>
<tr>
<td>4 Bands</td>
<td>Digit</td>
<td>Digit</td>
<td>Exponent</td>
<td>Tolerance</td>
<td></td>
</tr>
<tr>
<td>5 Bands</td>
<td>Digit</td>
<td>Digit</td>
<td>Exponent</td>
<td>Tolerance</td>
<td>Tolerance after 1000 hours</td>
</tr>
</tbody>
</table>
3 Band resistors have no band for the tolerance because it is assumed to be 20% of the nominal values. The fifth band is not an industry standard, but quite often it means the tolerance after 1000 hours of continuous use.

\[ R = AB \cdot 10^C, \quad \Delta R = R \cdot D \]

The bands are counted from left right. The following table reports the coding of the values using colors and a mnemonic sentence to remember the color code table.

<table>
<thead>
<tr>
<th>Color</th>
<th>Exponent</th>
<th>Tolerance (%)</th>
<th>Tolerance (%)</th>
<th>5th Band</th>
</tr>
</thead>
<tbody>
<tr>
<td>Big</td>
<td>Black</td>
<td>0</td>
<td>20</td>
<td>1%</td>
</tr>
<tr>
<td>Bart</td>
<td>Brown</td>
<td>1</td>
<td>1</td>
<td>0.1%</td>
</tr>
<tr>
<td>Rides</td>
<td>Red</td>
<td>2</td>
<td>2</td>
<td>0.01%</td>
</tr>
<tr>
<td>Over</td>
<td>Orange</td>
<td>3</td>
<td></td>
<td>0.001%</td>
</tr>
<tr>
<td>Your</td>
<td>Yellow</td>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Grave</td>
<td>Green</td>
<td>5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Blasting</td>
<td>Blue</td>
<td>6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Violent</td>
<td>Violet</td>
<td>7</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Guns</td>
<td>Gray</td>
<td>8</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Wildly.</td>
<td>White</td>
<td>9</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Go</td>
<td>Gold</td>
<td>-1</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>Shoot (him?)</td>
<td>Silver</td>
<td>-2</td>
<td>10</td>
<td></td>
</tr>
</tbody>
</table>

For example, the nominal resistance of a 4 band resistor having the sequence brown, black, orange and gold is

\[ R_{\text{nom.}} = 10k\Omega \]

\[ \Delta R_{\text{nom.}} = 5\%10k\Omega \]

\[ \Rightarrow R_{\text{nom.}} = (10.0 \pm 0.5)k\Omega \]

Resistor size (volume) is related to the power dissipation capability. Typical used values are 1/4W, 1/2W, 1W.
CHAPTER 4. DIRECT CURRENT NETWORK THEORY

4.9 First Laboratory Week

Sections 1, 2, 3, 5, and 7 of this chapter are required to complete the first laboratory week. A particular attention deserves the Thévenin equivalent circuit, which is the main topic of the experiment.

To experimentally study the basics of electronic networks, which is the scope of this laboratory week, we will use the following instruments:

- a digital multimeter (DMM) to measure voltage differences, currents, resistances

- a data acquisition system (DAQ),

- a voltage source.

Whenever you work with electronic circuits as a beginner (all ph3 students are considered beginners it doesn’t matter which personal skills they already have), some extra precautions must be taken to avoid injuries. These are the main ones:

- **Never connect instrument probes or leads to the power line or to an outlet.**

- **Do not try to fix/improve an instrument by yourself.**

- **Do not power up an instrument which is not working or disassembled.**

- **Do not touch a disassembled or partially disassembled instrument even if it is not powered.**

- **Wear protective goggles every time you use a soldering iron.**

- **To avoid explosions, never use a soldering iron on a powered circuit and batteries.**

- **Place a fan to disperse solder vapors for long period of soldering work.**
4.9. **FIRST LABORATORY WEEK**

### 4.9.1 Pre-Laboratory Problems

1. Find the current $I$ in the circuit shown below and confirm the two Kirchhoff’s Laws

![Circuit Diagram](image)

2. Find the equivalent Thévenin circuit for the following circuit respect to the point A and B:

![Circuit Diagram](image)

3. A voltage generator with voltage $V_0 = 10V$ and internal resistance $R_0 = 10\Omega$ can deliver a maximum current $I_0 = 50mA$. Using a series of 4 resistors, we want to produce the voltage differences $V_1 = 9V$, $V_2 = 5V$ and $V_3 = 3V$ measured from the negative pole of the generator and using the maximum deliverable current. Calculate the resistances of the 4 resistors.

4. A DAQ system uses a 10bit Analog to Digital Converter (ADC) with a dynamic range from 0V to 5.115V. What is the resolution $\Delta V$ and
the statistical uncertainty $\sigma_V$ of the ADC? Assume that the converted voltage follows the uniform statistical distribution.

### 4.9.2 Procedure

To assemble your circuit, you will use a so called solder-less breadboard which allows to connect electronic components by just plugging their leads into the holes of the board. The electrical connections of the solder-less board are show below:

Read completely the procedure before starting the circuit assembly, and taking measurements to avoid repeating parts of the procedure.

**Ohm's and Kirchhoff's Laws**

Set up the circuit shown below, using the nominal values of the resistors
Using a DMM or the DAQ system, do the following points:

1. Measure the resistances $R_0$, $R_1$, $R_2$, $R_3$, and the resistance $R_{tot}$ seen from the points C and D. Verify the resistance series and parallel laws for $R_{tot}$. No error analysis is required.

2. Verify Kirchhoff’s voltage law for a loop containing the voltage source. No error analysis is required.

3. Verify Kirchhoff’s current law for one of the nodes. No error analysis is required.

4. Using the DAQ system, the matlab program P9111GUISweepVoltage\(^1\), and varying the voltage $V$, verify Ohm’s law for the resistor $R_1$. Obtain the value of $R_1$ fitting all the collected of data-points (hint: compute the current flowing through $R_1$ using $R_0$).

5. Connect a resistive load with resistance $R_L$ “ad libitum” between A and B, and measure the voltage difference across it.

### Thévenin Equivalent Circuit

Using the previous circuit do the following points:

1. Using a proper connection of the resistors of your circuit, build the Thévenin equivalent circuit seen from terminals A and B.

2. Verify the equivalence with the previous circuit by connecting the same resistive load $R_L$ you used before.

### Ideal Voltage Source

Using the available voltage source do the following points:

1. Determine the internal resistance of the source.

2. Find for which interval of the resistive load, the voltage source is ideal within 5% (i.e. the maximum voltage difference across the load 5% of voltage difference with no load).

\(^1\)A way to avoid typing the entire matlab command is to type “P9”, and the tab key once or twice to get the list of all the command starting with “P9”
4.10 Second Laboratory Week

Sections 4, and 6 of this chapter are required to complete the first laboratory week.

Depending on the application, semiconductor diodes can have quite different parameters. Silicon diodes used in the laboratory typically have \( \eta \approx 2 \) for currents below \( \sim 100\text{mA} \), breakdown voltage \( V_b \approx -50\text{V} \), and reverse saturation current \( I_0 \sim 10\mu\text{A} \).

4.10.1 Pre-Laboratory Problems

1. Consider the RC circuit made of the series of a capacitor with capacitance \( C \) and a resistor of resistance \( R \). Demonstrate that voltage \( V(t) \) across capacitor, which is discharged through the resistor satisfies the following equation

\[
V(t) = V_0 e^{-t/\tau}, \quad \tau = RC,
\]

where \( V_0 \) is the initial voltage.

2. Choose the values of \( R \) and \( C \), with \( R \ll 1\text{M} \Omega \) to get the time constant to be \( \tau = 1\text{s} \). Prove that \( \tau \) has the dimension of a time (hint: use eq.(4.2) and (4.1)).

3. With \( \tau = 1\mu\text{s} \) determine the time \( t^* \) for the voltage \( V(t) \) to become less than 1% of the initial value \( V_0 \). Supposing that the sampling rate of a data acquisition system is 1000 samples/s, what is the minimum value of \( \tau \) necessary to measure a variation of 1% of \( V(t) \)?

4. In semiconductor materials like germanium and silicon the number of charge carriers \( n \), strongly depends on the absolute temperature \( T \),
i.e. \( n(T) = n_0 e^{-E_g/k_b T} \), where \( k_b = 8.6 \cdot 10^{-5} \text{eV/K} \) is the Boltzmann constant and \( E_g \) is the gap from the conduction band and the valence band. Given \( E_g = 0.67 \text{eV} \) for germanium, calculate the ratio of \( n \) at body temp to \( n \) at room temp. If \( R \propto 1/n \), what is the ratio of the \( R \) at the two temperatures? Repeat for silicon where \( E_g = 1.1 \text{eV} \).

### 4.10.2 Procedure

**RC Circuit Time Constant**

1. Determine the time constant \( \tau \) of the RC circuit using the following set-up, the data acquisition system and the matlab program P9111GUIDAQ

   ![RC Circuit Diagram](image)

   Choose \( R \) and \( C \) such that \( \tau \) is about 1 s. The uncertainty on the time is considered negligible compared to the uncertainty on the voltage.

2. Redo the time constant \( \tau \) measurement for a charging capacitor.

3. Compare the values of \( \tau \) obtained with the RC value obtained measuring directly the resistance and the capacitor.

**Semiconductor Diodes**

Build the following circuit, choosing the value of the resistor to limit the maximum current to \( \sim 5 \text{mA} \), and answer the next points:
1. Determine the polarity and the turn on voltage of a germanium diode and a silicon diode.

2. Determine the Boltzmann constant in eV (electron-volt) units measuring the V-I characteristic of a silicon diode with the matlab program P9111GUISweepVoltage. Assume the parameter $\eta_{si} = 2.0 \pm 0.1$ for the silicon diode.

3. Assuming the previous measured value of the Boltzmann constant determine the parameter $\eta_{ge}$ of a germanium diode.

4. Calculate the value of $\eta_{ge}$ for the germanium diode considering the standard value of $k \simeq 1.381 \cdot 10^{-23}$ J/K, and then compare it with the previous measured value of $\eta_{ge}$.
Chapter 5

Alternating Current Network Theory

In this chapter we will study the properties of electronic networks propagating sinusoidal currents (alternate currents/AC). In this case, voltage or current sources produce sinusoidal waves whose frequency can be ideally changed from 0 to $\infty$. The AC analysis of such circuits is valid once the network is at the steady state, i.e. when the transient behavior (such as that produced by closing or opening switches) is extinguished.

In general, if we have a sinusoidal signal (voltage, or current) applied to a circuit having at least one input and one output, we will expect a change in amplitude and phase at the output. The determination of these quantities for quite simple circuits can be very complex. It is indeed important to develop a convenient representation of sinusoidal signals to simplify the analysis of circuits in the AC regime.

5.1 Symbolic Representation of a Sinusoidal Signals, Phasors

A sinusoidal quantity (a sinusoidal current or voltage for example),
\[ A(t) = A_0 \sin(\omega t + \varphi), \]
is completely characterized by the amplitude $A_0$, the angular frequency $\omega$, and the initial phase $\varphi$. The phase $\varphi$ corresponds to a given time shift $t^*$ of the sinusoid ($\omega t^* = \varphi \Rightarrow t^* = \varphi / \omega$).
CHAPTER 5. ALTERNATING CURRENT NETWORK THEORY

Figure 5.1: Sinusoidal quantity $A(t)$ and its phasor representation $\vec{A}$ at the initial time $t = 0$ and at time $t$.

We can indeed associate to $A(t)$ an applied vector $\vec{A}$ of the complex plane with modulus $|\vec{A}| = A_0 \geq 0$, rotating counter-clock wise around the origin $O$, with angular frequency $\omega$, and initial angle $\phi$ (see figure 5.1). This vector is called phasor.

The complex representation of the phasor is indeed

$$\vec{A} = A_0 e^{j(\omega t + \phi)}, \quad j = \sqrt{-1},$$

or

$$\vec{A} = x + jy, \quad \left\{ \begin{array}{l}
    x = A_0 \cos(\omega t + \phi) \\
    y = A_0 \sin(\omega t + \phi)
\end{array} \right.$$

Extracting the real and the imaginary part of the phasor, we can easily compute its amplitude $A_0$ and phase $\phi$, i.e.

$$|\vec{A}| = \sqrt{\Re[\vec{A}]^2 + \Im[\vec{A}]^2} \quad \phi = \arg[\vec{A}] = \arctan \left( \frac{\Im[\vec{A}]}{\Re[\vec{A}]} \right), \quad (t = 0)$$

and reconstruct the real sinusoidal quantity. It is worthwhile to notice that in general, amplitude $A_0$ and phase $\phi$ are functions of the frequency.

The convenience of this representation will be evident, once we consider the operation of derivation and integration of a phasor.

\(^1\)To avoid confusion with the symbol of the electric current $i$, it is convenient to use the symbol $j$ for the imaginary unit.
5.2. CURRENT VOLTAGE EQUATION FOR PASSIVE IDEAL COMPONENTS WITH PHASORS

5.1. Derivative of a Phasor

Computing the derivative of a phasor $\vec{A}$, we get

$$\frac{d\vec{A}}{dt} = j\omega A_0 e^{j(\omega t + \phi)} = j\omega \vec{A},$$

i.e. the derivative of a phasor is equal to the phasor times $j\omega$.

5.1.2 Integral of a Phasor

The integral of a phasor $\vec{A}$ is

$$\int_{t_0}^{t} \vec{A} dt' = \frac{1}{j\omega} \left[ A_0 e^{j(\omega t + \phi)} - A_0 e^{j(\omega t_0 + \phi)} \right] = \frac{1}{j\omega} \vec{A} + \text{const.},$$

i.e. the integral of a phasor is equal to the phasor divided by $j\omega$ plus a constant. For the AC regime we can assume the constant to be equal to zero without loss of generality.

Symbols for ideal sinusoidal voltage and current generators are shown in figure 5.2.

---

5.2. CURRENT VOLTAGE EQUATION FOR PASSIVE IDEAL COMPONENTS WITH PHASORS

Let’s rewrite the I-V characteristic for the passive ideal components using the phasor notation. For sake of simplicity, we remove the arrow above the
phasor symbol. To avoid ambiguity, we will use upper case letters to indicate phasors, and lower case letters to indicate a generic time dependent signal.

5.2.1 The Resistor

For time dependent signals, Ohm’s law for a resistor with resistance $R$ is

$$v(t) = Ri(t).$$

Introducing the phasor $I = I_0e^{j\omega t}$ (see figure above), we get

$$v(t) = R I_0 e^{j\omega t},$$

and in the phasor notation

$$V = R I.$$  

Note that in this case, the frequency and time dependence is implicitly contained in the phasor current $I$.

5.2.2 The Capacitor

The variation of the voltage difference $dv$ across a capacitor with capacitance $C$, and due to the amount of charge $dQ$, is

$$dv = \frac{dQ}{C}. $$

If the variation happens in a time $dt$, and considering that

$$i(t) = \frac{dQ}{dt},$$
5.2. CURRENT VOLTAGE EQUATION FOR PASSIVE IDEAL COMPONENTS WITH PHASORS

we will have

\[ \frac{dv(t)}{dt} = \frac{1}{C} i(t), \quad \Rightarrow \quad v(t) = \frac{1}{C} \int_0^t i(t') dt' + v(0). \]

Introducing the phasor \( I = I_0 e^{j\omega t} \) (see figure above), we get

\[ v(t) = \frac{1}{C} \int_0^t I_0 e^{j\omega t'} dt' + v(0), \]

Using the phasor notation and supposing that for \( t = 0 \) the capacitor is discharged, we finally get

\[ V = \frac{1}{j\omega C} I, \quad , v(0) = 0. \]

5.2.3 The Inductor

The induced voltage \( v(t) \) of an inductor with inductance \( L \), is

\[ v(t) = L \frac{di(t)}{dt}. \]
Introducing the phasor \( I = I_0 e^{j\omega t} \) (see figure above), we get
\[
\nu(t) = L \frac{d}{dt} I_0 e^{j\omega t},
\]
and in the phasor notation
\[
V = j\omega L I.
\]

5.3 The Impedance and Admittance Concept.

Let's consider a generic circuit with a port, whose voltage difference and current are respectively the phasors \( V = V_0 e^{j(\omega t + \phi)} \), and \( I = I_0 e^{j(\omega t + \psi)} \). The ratio \( Z \) between the voltage difference and the current
\[
Z(\omega) = \frac{V}{I} = \frac{V_0}{I_0} e^{j(\phi - \psi)}.
\]
is said to be the impedance of the circuit.

The inverse
\[
Y(\omega) = \frac{1}{Z(\omega)}
\]
is called the admittance of the circuit.

For example, considering the results of the previous subsection, the impedance for a resistor, a capacitor, and an inductor are respectively
\[
Z_R = R, \quad Z_C(\omega) = \frac{1}{j\omega C}, \quad Z_L(\omega) = j\omega L,
\]
and the admittances are
\[
Y_R = \frac{1}{R}, \quad Y_C(\omega) = j\omega C, \quad Y_L(\omega) = \frac{1}{j\omega L}.
\]

In general, the impedance and the admittance of a circuit port is a complex function, which depends on the angular frequency \( \omega \). Quite often they are graphically represented by plotting their magnitude and phase.
5.3.1 Impedance in Parallel and Series

It can be easily demonstrated that the same laws for the total resistance of a series or a parallel of resistors hold for the impedance

\[ Z_{\text{tot}} = Z_1 + Z_2 + ... + Z_N, \]  
(impedances in series)

\[ \frac{1}{Z_{\text{tot}}} = \frac{1}{Z_1} + \frac{1}{Z_2} + ... + \frac{1}{Z_N}, \]  
(impedances in parallel)

5.3.2 Ohm’s Law for Sinusoidal Regime

Thanks to the impedance concept, we can generalize Ohm’s law and write the fundamental equation (Ohm’s law for sinusoidal regime)

\[ V(\omega) = Z(\omega)I(\omega). \]

5.4 Two-port Network

A linear circuit with one input and one output is called a two-port network (see figure 5.3).

To characterize the behavior of a two-port network, we can study the response of the output \( V_o \) as a function of the angular frequency \( \omega \) of a sinusoidal input \( V_i \).

In general, we can write

\[ V_o(\omega) = H(\omega)V_i(\omega), \quad \text{or} \quad H(\omega) = \frac{V_o(\omega)}{V_i(\omega)}. \]
where the complex function \( H(\omega) \) is called the \textit{transfer function of the two-port network}. The transfer function contains the information of how the amplitude and the phase of the input changes when it reaches the output. Knowing the transfer function of a two-port network, we characterize completely the circuit\(^2\). The definition of \( H(\omega) \) suggests the way of measuring the transfer function. In fact, exciting the input with a sinusoidal wave we can measure the amplitude and the phase lead or lag respect to the input of the output signal.

To graphically represent \( H(\omega) \), it is common practice to plot the magnitude \( |H(\omega)| \) in a double logarithmic scale, and the phase \( \arg[H(\omega)] \) in a semi-logarithmic scale for the angular frequency.

It is important to notice that it is not necessary to have an ideal sinusoidal generator to make the transfer function measurement. In fact, if the input amplitude changes with the frequency the ratio between the output will not change. The same is true for the phase, i.e. if the input phase changes the difference with the output phase cannot change.

Let’s study three common two port networks, the RC low-pass filter, the RC high-pass filter and the LCR series resonant circuit.

---

\(^2\)A much deeper understanding of the transfer function requires the concept of the Fourier transform and the Laplace transform.
5.4. TWO-PORT NETWORK

5.4.1 The RC Low-Pass Filter

Figure 5.4 shows the RC low-pass filter circuit. The input and the output voltage differences are respectively

\[ V_{\text{in}} = Z_{\text{in}} I = \left( R + \frac{1}{j\omega C} \right) I, \]
\[ V_{\text{out}} = Z_{\text{out}} I = \frac{1}{j\omega C} I, \]

and the transfer function is indeed

\[ H(\omega) = \frac{V_{\text{out}}}{V_{\text{in}}} = \frac{1}{1 + j\tau \omega}, \quad \tau = RC. \]

or

\[ H(\omega) = \frac{1}{1 + j\omega/\omega_0}, \quad \omega_0 = \frac{1}{RC}. \]

---

3\( V_{\text{out}} \) as function of \( V_{\text{in}} \) can be directly calculated using the voltage divider equation.
Computing the magnitude and phase of $H(\omega)$, we obtain

$$|H(\omega)| = \frac{1}{\sqrt{1 + \tau^2\omega^2}}$$

$$\arg(H(\omega)) = -\arctan\left(\frac{\omega}{\omega_0}\right)$$

Figure 5.5 shows the magnitude and phase of $H(\omega)$. The parameter $\tau$ and $\omega_0$ are called respectively the time constant and the angular cut-off frequency of the circuit. The cut-off frequency is the frequency where the output $V_{out}$ is attenuated by a factor $1/\sqrt{2}$.

It is worthwhile to analyze the qualitative behavior of the capacitor voltage difference $V_{out}$ at very low frequency and at very high frequency.

For very low frequency the capacitor is an open circuit and $V_{out}$ is essentially equal to $V_{in}$. For high frequency the capacitor acts like a short circuit and $V_{out}$ goes to zero.

The capacitor produces also a delay as shown in the phase plot. At very low frequency the $V_{out}$ follows $V_{in}$ (they have the same phase). The output $V_{out}$ loses phase ($\omega t = \phi \Rightarrow t = \phi/\omega$) when the frequency increases the frequency $V_{out}$ starts lagging due to the negative phase, $\phi$, and then reaches a maximum delay due to a phase shift of $-\pi/2$.

### 5.4.2 The CR High-Pass Filter

Figure 5.6 shows the CR high-pass filter circuit. The input and the output voltage differences are respectively

$$V_{in} = Z_{in}I = \left(R + \frac{1}{j\omega C}\right)I,$$

$$V_{out} = Z_{out}I = RI,$$

and indeed the transfer function is

$$H(\omega) = \frac{V_{out}}{V_{in}} = \frac{j\omega\tau}{1 + j\tau\omega}, \quad \tau = RC,$$

or

$$H(\omega) = \frac{j\omega/\omega_0}{1 + j\omega/\omega_0}, \quad \omega_0 = \frac{1}{RC}.$$
Computing the magnitude and phase of $H(\omega)$, we obtain

$$|H(\omega)| = \frac{\tau \omega}{\sqrt{1 + \tau^2 \omega^2}},$$

$$\arg(H(\omega)) = \arctan\left(\frac{\omega_0}{\omega}\right).$$

Figure 5.7 shows the magnitude and phase of $H(\omega)$. The definitions in the previous subsection for $\tau$, and $\omega_0$ hold for the RC high-pass filter.

Figure 5.7: CR high pass filter circuit transfer function.
5.4.3 The LCR Series Resonant Circuit

Figure 5.8 shows the LCR series circuit. Considering the voltage difference across the capacitor as the circuit output, we will have

\[ V_{in} = \left( R + j\omega L + \frac{1}{j\omega C} \right) I, \]

\[ V_{out} = \frac{1}{j\omega C} I, \]

and the transfer function will be

\[ H_C(\omega) = \frac{1}{j\omega RC - \omega^2 LC + 1}. \]

Computing the magnitude and phase of \( H(\omega) \), we obtain

\[ |H_C(\omega)| = \frac{1}{\sqrt{(1 - \omega^2 LC)^2 + (\omega RC)^2}}, \]

\[ \arg[H_C(\omega)] = \arctan \left( \frac{\omega RC}{\omega^2 LC - 1} \right). \]

For sake of simplicity It is convenient to define the two following quantities

\[ \omega_0^2 = \frac{1}{LC}, \quad Q = \frac{1}{R} \sqrt{\frac{L}{C}}. \]
The parameter \( Q \) is called the quality factor of the circuit. Considering the previous definitions, and after some algebra \( H_C(\omega) \) can be rewritten as

\[
H_C(\omega) = \frac{\omega_0^2}{-\omega^2 + j\omega \frac{\omega_0}{Q} + \omega_0^2}.
\]  

(5.1)

The magnitude has an absolute maximum for

\[
\omega_C^2 = \omega_0^2 \left(1 - \frac{1}{2Q^2}\right),
\]

(angular resonant frequency)

and the maximum is

\[
|H_C(\omega_C)| = \frac{Q}{\sqrt{1 - \frac{1}{4Q^2}}}, \quad \text{if } Q \gg 1 \Rightarrow \omega_C \approx \omega_0, \quad |H_C(\omega_C)| \approx Q
\]

Far from resonance the approximate behavior of \( |H_C(\omega)| \) is

\[
\begin{align*}
\omega &\ll \omega_C \quad \Rightarrow \quad |H_C(\omega)| \approx 1 \\
\omega &\gg \omega_C \quad \Rightarrow \quad |H_C(\omega)| \approx \frac{\omega_0^2}{\omega^2}
\end{align*}
\]

Figure 5.9 shows the magnitude and phase of \( H_C(\omega) \).
Figure 5.9: Transfer Function $H_C(\omega)$ of the LCR series resonant circuit.
5.5 First Laboratory Week

5.5.1 Pre-laboratory Exercises

1. Calculate the total impedance of a series of a resistor with a capacitor and for a parallel of a resistor with a capacitor. Do your results confirm the statement that capacitors behave as a short circuit at high frequencies and as an open circuit at low frequencies?

2. Calculate the magnitude of the total impedance for a series of a resistor with a capacitor having $R = 10 \, \text{k}\Omega$, $C = 2.5 \, \text{nF}$, and $\nu = 20 \, \text{kHz}$. Calculate the same quantity for a parallel of a resistor with a capacitor having $R = 10 \, \text{M}\Omega$, $C = 30 \, \text{pF}$, and $\nu = 20 \, \text{kHz}$.

3. The circuit shown below includes the impedance of the input channel of the CRT oscilloscope, and $V_s$ is indeed the real voltage measured by the instrument.

![Circuit Diagram]

Find the voltage $V_s(\omega)$, and the angular cut-off frequency $\omega_0$ of the transfer function $V_s/V_{in}$, i.e. the value of $\omega$ for which $|V_s/V_{in}|$ is $1/\sqrt{2}$ of its DC value. It is convenient to compute the impedance $Z$ highlighted in the box first and then use the voltage divider equation to write $V_s(\omega)$.

Show that for $\omega = 0$ the $V_s(\omega)$ formula simplifies and becomes the resistive voltage divider equation.

Demonstrate that the conditions to neglect the input impedance of the oscilloscope are the following:

$$ C \gg C_s, \quad R \ll R_s $$

4. Considering the previous circuit, calculate the value of $R$ to obtain $V_{in} \simeq V_s$ with a fractional systematic error of 1%, if $\omega = 0 \, \text{rad/s}$ and $R_s = 1\, \text{M}\Omega$. 
5. Read the first two sections of the oscilloscope notes (see appendix C).

6. Considering the figure below (a “snapshot” of an oscilloscope display), determine the peak to peak amplitude, the DC offset, the frequency of the two sinusoidal curves, and the phase shift between the two curves (channels horizontal axis position is indicated by an arrow and the channel name on the right of the figure).

5.5.2 Procedure

Read carefully the text before starting the laboratory measurements.

Note that instead of the angular frequency $\omega$, this procedure reports the frequency $\nu (\omega = 2\pi\nu)$, which is more often used in laboratory measurements.

“BNC” cables and wires terminated with “banana” connectors are available to connect circuits to the available instruments.
BNC\textsuperscript{4} cables, a diffused type of radio frequency (RF) coaxial cable, have an intrinsic capacitance due to their geometry as shown in the figure below.

They have typical linear density capacitance $\Delta C/\Delta l \sim 100$ pF/m. Single wires have usually smaller capacitance than BNC cables. Unfortunately, their capacitance strongly depends on how they are positioned one respect to the others.

- Build a RC low-pass filter or a CR high-pass filter with a cut-off frequency $\nu_0$ between 1 kHz and 100 kHz, using values for $R$ and $C$, which makes the input impedance of the oscilloscope $Z_s$ negligible compared to the impedance of your circuit.

- Using the oscilloscope, experimentally find $|H(\nu_0)|$ and arg $[H(\nu_0)]$ (at the cut-off frequency $\nu_0$), and compare them with the theoretical data using the measured values of $R$ and $C$.

- Plot the transfer function $H(\nu)$ of your circuit by measuring $|H(\nu)|$, and arg $[H(\nu)]$. Measure at least 10 data points in a frequency range one decade above and below your cut-off frequency. Then, fit the experimental data with the proper theoretical curves. Use the appropriate fitting functions (FitLowPass1stOrderMag, FitLowPass1stOrderPhase, FitHighPass1stOrderMag, FitHighPass1stOrderPhase.)

- Build a RC low-pass filter using a capacitance $C$ comparable with the input capacitance $C_s$ of the oscilloscope or with the coaxial BNC cable capacitance $C_{BNC}$ you are using. Check if the perturbation induced by the instrument at the cut-off frequency $\nu_0$ can be explained by the added capacitances $C_s$ and $C_{BNC}$.

\textsuperscript{4}“BNC” seems to stand for Bayonet Neill Concelman (named after Amphenol engineer Carl Concelman). Other sources claim that the acronym means British Navy Connector. What is certain is that the BNC connector was developed in the late 1940’s as a miniature version of the type C connector (what does the “C” stand for?)
5.6 Second Laboratory Week

5.6.1 Pre-laboratory Exercises

1. Considering an inductor made of a solenoid with large inductance \((L \sim 10\text{mH})\), resistance \(R_L = 80\Omega\), wire diameter \(d = 100\mu\text{m}\), and resistivity of \(\rho \approx 16\text{n} \Omega \cdot \text{m} \) (copper), determine the length \(l\) of the coil wire.

2. Demonstrate that the magnitude of the LCR series transfer function \(H_C(\omega)\) has a maximum for \(\omega = \omega_C\) and the maximum is equal to the quality factor \(Q\), \((Q \gg 1)\).

3. Determine the capacitance \(C\) for a LCR series circuit to have a resonant frequency \(\nu_C = 5\text{kHz}\) if \(L = 10\text{mH}\), and \(R = 15\Omega\). Then, calculate the quality factor of the LCR series circuit.

4. Determine the total impedance \(Z(\nu)\) of the LCR series circuit. Using the components values of problem 3, plot from 100 Hz to 50 kHz magnitude and phase of \(Z\) using logarithmic scales. Use the imaginary unit "\(1i\)" to compute the impedance \(Z\) and functions \(\text{abs}\) and \(\text{angle}\) to extract its magnitude and phase. Consult the matlab script examples posted on the ph3 web pages.

5. Using a better model of the inductor below and a capacitance value \(C_p = 10\text{nF}\)

![Diagram of LCR circuit with inductor, resistor, and capacitor]

determine the total impedance \(Z(\nu)\) of the LCR series circuit, and redo the logarithmic plot of the magnitude and phase of the total circuit impedance \(Z\).
5.6.2 Procedure

- Construct a LCR series resonant circuit shown above with a resonant frequency $\nu_C$ between 5 kHz to 10 kHz using a 10 mH inductor. Note that the resistance $R_s$ is the internal resistance of the function generator, and the resistance $R_L$ is the resistance of the inductor. Use polyester capacitors to make the circuit. Ask your instructor for the right type of capacitors.

- To verify that the circuit is properly working, use the oscilloscope and vary the frequency to find $\nu_C$.

- Measure the magnitude and phase of the transfer function $H_C(\nu)$ using the data acquisition program P9111GUISweepSineTF. Connect the data acquisition inputs $A_0$ and $A_1$ to measure the LCR circuit input and the output respectively (see previous figure). Fit the magnitude and phase of $H_C$ to determine the frequency $\nu_0$ and the quality factor $Q$. To associate an uncertainty to the fit parameters, measure and fit the transfer function 5 times. To automate the analysis, use a matlab script with a “for loop”. Ask your instructor about the script if you need help.

- Using the past measured values of $R_L$, $L$, and $C$, calculate $\nu_0$ and $Q$ and compare them with the values obtained from the fit.

- To verify how well the inductor and the capacitor behave as ideal components, measure their total series impedance $Z(\nu) = V(\nu)/I(\nu)$ using the data acquisition program P9111GUISweepSineTF and the
circuit below. Chose a new capacitance value to lower \( \nu_C \) down to about 1 kHz.

![Circuit Diagram]

To measure the current, place a 1 k\( \Omega \) resistor in series with the inductor and the capacitor between the inductor and the function generator as shown in the figure above. Connect the data acquisition inputs \( A_0 \) and \( A_1 \) to measure the voltage across the entire components series and across the inductor and capacitor series.

- Fit the magnitude \( |Z(\nu)| \) of the impedance \( Z(\nu) \) to your experimental data. Use the following script to fit your experimental data and note down any discrepancies. The script assumes that the vector \( f \) contains the frequency and the vector \( Zm \) contains the impedance calculated from the measured values.

```matlab
RL = ?; % Ohms, Inductor's resistance
L = ?; % H, Inductor's inductance
C = ?; % F, Capacitor's capacitance
S.sf = 'abs(RL + 1./(1i*w*C) + 1i*w*L)'; % fitting function
S.ParName = {'RL', 'L', 'C'};
S.Variable = {'w', '|Z|'};
S.x = 2*pi*f;
S.y = abs(Zm);
S.p = [RL, L, C]; % Initial parameters values
S.PlotType = 'loglog';
S = FitCurve(S);
```

- To improve the agreement with the experimental data, consider the better model of the inductor introduced in the pre-lab problems. Modify the previous script to account for the better inductor model.
(different fit function and an extra parameter $C_p$, with an extra initial value) and fit the data to the new theoretical curve of $|Z(\nu)|$. Use 10 nF for the initial value of the capacitance $C_p$.

- Does the better inductor’s model improve the agreement with the experimental data or there are still some discrepancies?
CHAPTER 5. ALTERNATING CURRENT NETWORK THEORY
Appendix A

Measurements and Significant Figures (Draft)

The purpose of this appendix is to explain how to properly report a measurement and its uncertainty based on the number of significant figures of the uncertainty. This is a very important task to perform because measurements are meaningless without their associated uncertainty.

The significant figures (or significant digits) of a number are the digits necessary to specify our knowledge of that number’s precision, and therefore they must be carefully chosen to accurately represent that precision. Let’s familiarize with first some concepts about numbers representation.

A.1 Scientific Notation

A number $n$ written in scientific notation has the form

$$n = \pm x.xxx \cdot 10^{\pm yyy}$$

where $x$, $y$, and $z$ are an arbitrary number of digits.

$\pm x.xxx$ is the mantissa

$\pm yyy$ is the exponent

The mantissa is always written with exactly one non-zero digit to the left of the decimal point. For example,

$$n = 12.345 \quad \Rightarrow \quad n = 1.2345 \cdot 10^{-1}.$$
A.2 Significant Figures

**Definition:** The digits in the mantissa of a number expressed in scientific notation are called significant figures or significant digits. A zero is always significant.

To determine the number of significant figures, rewrite the number in scientific notation and count the digits in the mantissa. Let’s consider some examples.

<table>
<thead>
<tr>
<th>Number</th>
<th>Scientific Notation</th>
<th># Significant Figures</th>
</tr>
</thead>
<tbody>
<tr>
<td>134.00</td>
<td>$1.3400 \cdot 10^2$</td>
<td>5</td>
</tr>
<tr>
<td>0.01023</td>
<td>$1.023 \cdot 10^{-2}$</td>
<td>4</td>
</tr>
<tr>
<td>2.3E-4</td>
<td>$2.3 \cdot 10^{-4}$</td>
<td>2</td>
</tr>
<tr>
<td>1.2009</td>
<td>$1.2009 \cdot 10^0$</td>
<td>5</td>
</tr>
<tr>
<td>−0.03201450</td>
<td>$-3.201450 \cdot 10^2$</td>
<td>7</td>
</tr>
<tr>
<td>20, 150</td>
<td>$2.015 \cdot 10^4$</td>
<td>4</td>
</tr>
<tr>
<td>$\Rightarrow$</td>
<td>$2.0150 \cdot 10^4$</td>
<td>5</td>
</tr>
</tbody>
</table>

The last example is ambiguous because trailing zeros in integer numbers may or may not be significant. Using scientific notation removes this ambiguity.

A.3 Significant Figures in Measurements

Scientific measurements should be reported according to the guidelines of the Bureau International des Poids et Mesures (BIPM). These guidelines may be summarized as follows.

- Every measurement should include its uncertainty and its units.
- A value and its uncertainty should have the same units, exponent, and number of significant figures.
- Scientific notation and SI units and prefixes should be used.

1. Scientific notation and SI units and prefixes should be used.

1 [http://www.bipm.org/en/home/]
A.3. SIGNIFICANT FIGURES IN MEASUREMENTS

- Units are symbols, not abbreviations, and do not need punctuation marks. The official symbol for each unit should be used. (e.g., “3 seconds” is written “3 s” rather than “3 s,” “3 sec.,” or “3 sec”)

For example, to report $c$, the speed of light in a vacuum, with its uncertainty we could choose one of the following equivalent options:

$$c = (2.99792 \pm 0.00030) \cdot 10^8 \text{ m/s} \quad \text{(extended notation)},$$
$$c = (0.299792 \pm 0.000030) \text{ Gm/s} \quad \text{(extended notation with unit prefix)},$$
$$c = 2.99792(30) \cdot 10^8 \text{ m/s} \quad \text{(concise notation}).$$

A.3.1 Significant Figures for Direct Measurements

Let’s consider the case of a single measurement made directly with an instrument. In this case, the measurement value and its uncertainty must each have enough digits to account for the instrument resolution. Instrument resolution is the maximum error that the instrument gives under the specified conditions, (e.g., measurement range, temperature, humidity, pressure, etc.). A statistical analysis and a better characterization of the instrument to reduce systematic errors can provide a smaller error estimation.

A.3.1.1 Length Measurement Example

Suppose we use a caliper to measure the outside diameter of a cylinder. The caliper manufacturer has specified a resolution $\Delta x = 0.05 \text{ mm}$ for the caliper. The manufacturer’s quoted resolution is usually the maximum error for the instrument when used properly. If a single measurement gives 12.15 mm, three proper ways to report the measurement are

$$x = (12.15 \pm 0.05) \text{ mm},$$
$$x = (1.215 \pm 0.005) \cdot 10^{-2} \text{ m},$$

and if you really like it

$$x = 1.215(5) \cdot 10^{-2} \text{ m}.$$ 

If we trust the quoted instrument resolution, it would be misleading to report a measurement as 12.12 mm. This would imply a measurement precision better than the instrument resolution.
Remember that the goal is to accurately estimate the measurement uncertainty. The quoted instrument resolution assumes proper measurement technique. If we are not careful, we may introduce errors that exceed the instrument’s quoted accuracy.

A.3.2 Significant Figures for Statistical Measurements

A statistical measurement is made of at least three numbers: the observed measured value, the uncertainty, and the confidence level. The confidence level is the probability, usually in percent, associated to the interval given by the observable value and the uncertainty. In other words, the confidence level is the probability of a new measurement to lie inside the mentioned interval. If the confidence level for a statistical measurement is omitted, it is standard to assume the $1\sigma$ interval of $68.3\%$.\(^2\)

Statistical measurements should be reported in extended or concise notation as follows.

\[
X = (x.xxx xxx \pm y.yyy yyy) \text{ units} \quad [zz.zz\% \text{ Confidence}],
\]

\[
X = x.xxx xxx (yyy) \text{ units} \quad [zz.zz\% \text{ Confidence}].
\]

Sometimes the uncertainty interval is asymmetric:

\[
X = (x.xxx xxx \pm \frac{y.yyy yyy}{w.www wwww}) \text{ units} \quad [zz.zz\% \text{ Confidence}],
\]

\[
X = x.xxx xxx (\frac{yyy}{w.www}) \text{ units} \quad [zz.zz\% \text{ Confidence}].
\]

How many significant figures should be quoted when reporting a statistical uncertainty? For a direct measurement, there was a clear upper bound from the instrument resolution. For a statistical measurement, there is no such generally-applicable standard. The number of significant figures in a reported result should be chosen to effectively convey the value.

\(^2\)68.3\% is the probability that a single measurement from a normal distribution lies within one standard deviation of the mean.
and its precision to the reader. Thus, the number should be large enough
to accurately reflect the precision, but not so large as to overwhelm the
reader with an endless string of digits.\footnote{There are varying rules of thumb for choosing an appropriate number of significant
figures based on the uncertainty of the measurement. Experimenters who find them-
selves horrified by the gruesome sight of too many digits in the uncertainty should be
tolerant because such “horrible” practice does not seem to inhibit scientific progress.}

A.3.2.1 Electron Mass Example

For example, the standard value for the electron’s mass is the statistical
combination of a number of measurements. We therefore report its value
as a statistical measurement:

\[
m_e = (0.510098910 \pm 0.000000013) \text{MeV,}
\]

\[
m_e = 0.510098910(13) \text{MeV.}
\]

In this case, no confidence interval has been specified. We therefore
assume the confidence level to be 68.3%, the $1\sigma$ interval.

A.3.3 Significant Digits in Calculations

The above discussion concerns reporting a measurement result. When per-
forming a calculation, \textit{do not} follow these guidelines for intermediate results—
keep as many digits as is practical to avoid rounding errors. At the end,
round the final answer to a reasonable number of significant figures for
presentation.

A.4 Unit Prefixes

In addition to units, the International System of Units (SI, from the French
“Système International”) specifies a set of standard prefixes that modify
the magnitude of a unit.\footnote{Detailed information about the SI, including prefixes as well as units, can be obtained
from the BIPM website at http://www.bipm.org/en/si/si_brochure/} The standard prefixes are listed in Table A.1 By
APPENDIX A. MEASUREMENTS AND SIGNIFICANT FIGURES (DRAFT)

<table>
<thead>
<tr>
<th>Magnitude</th>
<th>Symbol</th>
<th>Name</th>
<th>Magnitude</th>
<th>Symbol</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{-24}$</td>
<td>y</td>
<td>yocto</td>
<td>$10^{+1}$</td>
<td>da</td>
<td>deca</td>
</tr>
<tr>
<td>$10^{-21}$</td>
<td>z</td>
<td>zepto</td>
<td>$10^{+2}$</td>
<td>h</td>
<td>hecto</td>
</tr>
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<td>a</td>
<td>atto</td>
<td>$10^{+3}$</td>
<td>k</td>
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<td>nano</td>
<td>$10^{+12}$</td>
<td>T</td>
<td>tera</td>
</tr>
<tr>
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<td>$\mu$</td>
<td>micro</td>
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<td>$10^{+18}$</td>
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<td>c</td>
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<td>$10^{+21}$</td>
<td>Z</td>
<td>zetta</td>
</tr>
<tr>
<td>$10^{-1}$</td>
<td>d</td>
<td>deci</td>
<td>$10^{+24}$</td>
<td>Y</td>
<td>yotta</td>
</tr>
</tbody>
</table>

Table A.1: SI Prefixes

using a prefix, the exponent used to report a measurement in scientific notation can be modified but the number of significant figures is unchanged. For example,

\[ l = 0.00021 \text{m} = 2.1 \cdot 10^{-4} \text{m} = 2.1 \cdot 10^{-1} \text{mm} = 2.1 \cdot 10^{2} \mu\text{m}. \]

Although one should generally use standard SI units and prefixes, there are exceptions. In some contexts, non-standard units are more convenient for historical or pragmatic reasons. For example, astronomers customarily measure certain distances in parsecs.\(^5\) An astronomer working with distances to nearby stars would be well-advised to use this non-standard unit. When deciding between an SI and a non-standard unit, one should choose the option that most clearly communicates a result to the intended audience.

\(^5\)A parsec, or parallax-second, is the distance from the Sun to a star that has a parallax of 1 arc second from the Sun. In other words, if we build a right triangle with the Sun on its right angle vertex, the Earth and the start each one on the other vertices, and the angle of the star’s vertex is 1 arc second, then the distance star Sun is 1 parsec. The distance Earth Sun is 1 AU (Astronomical Unit) which is the estimated average Earth Sun distance. One parsec is approximately $3.086 \cdot 10^{16}$ m.
Appendix B

The Vernier

The vernier\(^1\) is a device that acts like a “mechanical scale magnifier”. It allows to read fractions \(1/n\) of the main scale divisions using the maximum possible resolution given by the technology used to manufacture the scale.

The vernier is essentially a pair of adjacent linear scales (see figure B.2) with one scale, the auxiliary scale, sliding parallel along the other scale, the main scale.

![Figure B.1: Example of a vernier with \(k = 5\), \(n = 4\), and \(\delta x = 1/4\) units. The two scales are aligned and the 4-th auxiliary division is aligned to the \(kn - 1 = 19\)-th division of the main scale.](image)

Each auxiliary scale division is \(k - 1/n\) shorter than \(k\) divisions of the main scale. This means that if we align the first auxiliary scale division to the \(k\)-th division of the main scale, the distance \(x\) between the origin of the two scales will be \(x = 1/n\) units. If we align the second division

---

\(^1\)Pierre Vernier XVII sec. mathematician and scientist inventor of the so called vernier caliper.
then \( x = \frac{2}{n} \) units. In general, aligning the \( m \)-th division of the auxiliary scale to the \( m \) times \( k \)-th division of the main scale, will make the distance between the origin of the two scales equal to \( x = \frac{m}{n} \).

### B.1 Measuring with a Vernier

If we perform a measurement of a distance \( x \) (see figure B.2) and we have

- the auxiliary scale zero after the \( N \)-th division of the main scale and
- the \( m \)-th auxiliary scale division is the division aligned to one of the main scale divisions, then we will have

\[
x = \left( N + \frac{m}{n} \right) \text{ units}.
\]

Because we cannot identify an interval were the measurement lies in, the read error \( \delta x \) to assign to \( x \) is

\[
\delta x = \frac{1}{n} \text{ units}.
\]

In the example of figure B.2, we have

\[
\begin{cases} 
  N = 8 \\
  m = 3 \\
  n = 4
\end{cases} \quad \Rightarrow \quad x = \left( 8 + \frac{3}{4} \right) = 8.75 = (8.8 \pm 0.3) \text{ units}.
\]

The \( 1/n \) factor and the units should be printed out on the vernier.
The resolution limit of a vernier, depends on the accuracy\(^2\) of inscribing both scales. If we have more than one coincident marks, the instrument claimed resolution is illusory.

Calipers are one of the most common instruments that uses a vernier.

### B.2 Probability Density Function Using a Vernier

Let’s suppose that we use an instrument with a vernier to measure a distance \(x\). In this case, we cannot assume that \(x\) follows a uniform PDF as we did for measurements performed using single linear scales. In fact, because we look for the best aligned divisions pair to perform the measurement, we cannot identify an interval were the measurements lies in. If we could, it means that the vernier is not properly designed or manufactured. In other words, the instrument is practically unusable at least with the resolution claimed by the manufacturer.

\(^2\)The vernier resolution depends mainly on the technology used to manufacture the two scales and the length of the main scale. In fact, the range limits the instrument resolution because it is hard to keep a given accuracy for an arbitrary length of the scale.
Appendix C

The Cathode Ray Tube Oscilloscope

Every time we need to analyze or measure an electronic signal in the time domain we will probably use some type of oscilloscope. The oscilloscope is therefore one of the most useful tools used in a laboratory. Practically, it is an indispensable instrument for measuring, designing, manufacturing, or repairing electronic equipment.

Quite often, one can still find old Cathode Ray Tube (CRT) oscilloscopes even in modern laboratory, mainly because of the inadequacy of state of the art digital oscilloscope scopes to represent very fast signals. It is therefore worthwhile to study this device and understand how a CRT works and also its limitations.

C.1 The Cathode Ray Tube Oscilloscope

The cathode ray tube oscilloscope is essentially an analog\(^1\) instrument that is able to measure time varying electric signals. It is made of the following functional parts (see figure C.1):

- the cathode ray tube (CRT),
- the trigger,

\(^1\)Hybrid instruments combining the characteristics of digital and analog oscilloscopes, with a CRT, are also commercially available.
• the horizontal input,
• the vertical input,
• time base generator.

Let’s study in more detail each component of the oscilloscope.

C.1.1 The Cathode Ray Tube

The CRT is a vacuum envelope hosting a device called an electron gun, capable of producing an electron beam, whose transverse position can be modulated by two electric signals (see figures C.1 and C.7).

When the electron gun cathode is heated by wire resistance because of the Joule effect it emits electrons. The increasing voltage differences between a set of shaped anodes and the cathode accelerates electrons to a terminal velocity $v_0$ creating the so called electron beam.

The beam then goes through two orthogonally mounted pairs of metallic plates. Applying voltage difference to those plates $V_x$ and $V_y$, the beam is deflected along two orthogonal directions ($x$ and $y$) perpendicular to its direction $z$. The deflected electrons will hit a plane screen perpendicular to the beam and coated with florescent layer. The electrons interaction with this layer generates photons, making the beam position visible on the screen.

C.1.2 The Horizontal and Vertical Inputs

The vertical and horizontal plates are independently driven by a variable gain amplifier to adapt the signals $v_x(t)$, and $v_y(t)$ to the screen range. A DC offset can be added to each input to position the signals on the screen. These two channels used to drive the signals to the plates signals are called horizontal and vertical inputs of the oscilloscope.

In this configuration the oscilloscope is an x-y plotter.

C.1.3 The Time base Generator

If we apply a sawtooth signal $V_x(t) = at$ to the horizontal input, the horizontal screen axis will be proportional to time $t$. In this case a signal $v_y(t)$
Figure C.1: Sketch of the functional parts of the analog oscilloscope, preamplifier, amplifier, trigger, time base generator, and CRT.
applied to the vertical input, will depict on the oscilloscope screen the signal time evolution.

The internal ramp signal is generated by the instrument with an amplification stage that allows changes in the gain factor $\alpha$ and the interval of time shown on the screen. This amplification stage and the ramp generator are called the *time base generator*.

In this configuration, the horizontal input is used as a second independent vertical input, allowing the plot of the time evolution of two signals.

Visualization of signal time evolution is the most common use of an oscilloscope.

Figure C.2: Periodic Signal triggering. The second trigger is ignored because the ramp is still of the sawtooth signal is still increasing to its maximum $\alpha T$.

### C.1.4 The Trigger

To study a periodic signal $v(t)$ with the oscilloscope, it is necessary to synchronize the horizontal ramp $V_x = at$ with the signal to obtain a steady plot of the periodic signal. The trigger is the electronic circuit which provides this function. Let’s qualitatively explain its behavior.

The trigger circuit compares $v(t)$ with a constant value and produces a pulse every time the two values are equal and the signal has a given
C.2. OSCILLOSCOPE INPUT IMPEDANCE

A good approximation of the input impedance of the oscilloscope is shown in the circuit of figure C.3. The different input coupling modes (DC AC GND) are also represented in the circuit.

The amplifying stage is modeled using an ideal amplifier (infinite input impedance) with a resistor and a capacitor in parallel to the amplifier input.

The switch allows to ground the amplifier input and indeed to vertically set the origin of the input signal (GND position), to directly couple the input signal (DC position), or to mainly remove the DC component of the input signal (AC position).

\[ C \]

\[ R \]

\[ C_s \]

Input DC AC GND

C

Pre-Amplifier

High Voltage Amplifier

Deflection Plates

Figure C.3: Oscilloscope input impedance representation using ideal components (gray box). Input channel coupling is also shown.

In general, the sawtooth signal period \( T \) and the period of \( v(t) \) are not equal.
C.3 Oscilloscope Probe

An oscilloscope probe is a device specifically designed to minimize the capacitive load and maximize the resistive load added when the instrument is connected to the circuit. The price to pay is an attenuation of the signal that reaches the oscilloscope input\(^3\).

Let’s analyze the behavior of a passive probe. Figure C.4 shows the equivalent circuit of a passive probe and of the input stage of an oscilloscope. The capacitance of the probe cable can be considered included in \(C_s\).

![Figure C.4: Oscilloscope input stage and passive probe schematics. The equivalent circuit made of ideal components for the probe shielded cable is not shown.](image)

Considering the voltage divider equation, we have

\[
H(j\omega) = \frac{V_s}{V_i} = \frac{Z_s}{Z_p + Z_s},
\]

where

\[
\frac{1}{Z_s} = j\omega C_s + \frac{1}{R_s}, \quad \frac{1}{Z_p} = j\omega C_p + \frac{1}{R_p},
\]

and then

\[
Z_s = \frac{R_s}{j\omega \tau_s + 1}, \quad Z_p = \frac{R_p}{j\omega \tau_p + 1}.
\]

\(^3\)Active probes can partially avoid this problems by amplifying the signal.
C.3. OSCILLOSCOPE PROBE

Defining the following parameters

$$\tau_p = C_p R_p, \quad \alpha = \frac{R_s}{R_s + R_p}, \quad \beta = \frac{C_p}{C_s + C_p},$$

and after some tedious algebra, equation (C.1) becomes

$$H(j\omega) = \alpha \frac{1 + j\omega \tau_p}{1 + j\omega \frac{\alpha}{\beta} \tau_p},$$

which is the transfer function from the probe input to the oscilloscope before the ideal amplification stage.

The DC and high frequency gain of the transfer function $H(j\omega)$ are respectively

$$H(0) = \alpha, \quad H(\infty) = \beta.$$

The numerator and denominator of $H(j\omega)$ are respectively equal to zero, (the zeros and poles of $H$) when

$$\omega = \omega_z = j \frac{1}{\tau_p}, \quad \omega = \omega_p = j \frac{\beta}{\alpha} \frac{1}{\tau_p}.$$

Figure C.5 shows the qualitative behavior of $H$ for $\frac{\alpha}{\beta} > 1$.

C.3.1 Probe Frequency Compensation

By tuning the variable capacitor $C_p$ of the probe, we can have three possible cases

$$\frac{\alpha}{\beta} < 1 \Rightarrow \text{over-compensation}$$
$$\frac{\alpha}{\beta} = 1 \Rightarrow \text{compensation}$$
$$\frac{\alpha}{\beta} > 1 \Rightarrow \text{under-compensation}$$

if $\alpha < \beta$ the transfer function attenuates more at frequencies above $\omega_z$, and the input signal $V_i$ is distorted.

if $\alpha = \beta$ the transfer function is constant and the input signal $V_i$ will be undistorted and attenuated by a factor $\alpha$. 
if $\alpha > \beta$ the transfer function attenuates more at frequencies below $\omega_p$ and the input signal $V_i$ is distorted.

The ideal case is indeed the compensated case, because we will have increased the oscilloscope input impedance by a factor $\alpha$ without distorting the signal.

The probe compensation can be tuned using a signal able to show a clear distortion when it is filtered. A square wave signal is very useful in this case because it shows a quite different distortion if the probe is under or over compensated. Figure C.6 sketches the expected square wave distortion for the two uncompensated cases.

It is important to notice that if

$$\frac{\alpha}{\beta} = 1, \quad \Rightarrow \frac{R_s}{R_p} = \frac{C_p}{C_s},$$

and this condition implies that:

- the voltage difference $V_1$ across $R_s$ is equal the voltage difference $V_2$ across $C_s$, i.e $V_1 = V_2$
C.3. OSCILLOSCOPE PROBE

- the voltage difference $V_3$ across $R_p$ is equal the voltage difference $V_4$ across $C_p$, i.e. $V_3 = V_4$

- and therefore $V_1 + V_2 = V_3 + V_4$.

This means that **no current is flowing through the branch AB when the probe is compensated**, and we can consider just the resistive branch of the circuit to calculate $V_s$. Applying the voltage divider equation, we finally get

$$V_s = \frac{R_s}{R_s + R} V_i$$

The capacitance of the oscilloscope does not affect the oscilloscope input anymore, and the oscilloscope+probe input impedance $R_i$ becomes greater, i.e.

$$R_i = R_s + R_p.$$ 

Figure C.6: Compensation of a passive probe using a square wave. Left figure shows an over compensated probe, where the low frequency content of the signal is attenuated. Right figure shows the under compensated case, where the high frequency content is attenuated.
C.4 Beam Trajectory

Let’s consider the electron motion through one pair of plates.

The electron terminal velocity $v_0$ coming out from the gun can be easily calculated considering that its initial potential energy is entirely converted into kinetic energy, i.e.

$$\frac{1}{2} \mu v_0^2 = eV_0, \quad \Rightarrow \quad v_0 = \sqrt{\frac{2eV_0}{\mu}},$$

where $\mu$ is the electron mass, $e$ the electron charge, and $V_0$ the voltage applied to the last anode.

If we apply a voltage $V_y$ to the plates whose distance is $h$, the electrons will feel a force $F_y = eE_y$ due to an electric field

$$|E_y| = \frac{V_y}{h}.$$

The equation of dynamics of the electron inside the plates is

$$\mu \ddot{z} = 0, \quad \Rightarrow \quad \dot{z} = v_0,$$

$$\mu \dot{y} = e|E_y|.$$

Figure C.7: CRT tube schematics. The electron enters into the electric field and makes a parabolic trajectory. After passing the electric field region it will have a vertical offset and deflection angle $\theta$. 
Supposing that $V_y$ is constant, the solution of the equation of motion will be

$$z(t) = \sqrt{\frac{2eV_0}{\mu}} t,$$

$$y(t) = \frac{1}{2} \frac{eV_y}{\mu h} t^2.$$

Removing the dependency on the time $t$, we will obtain the electron beam trajectory, i.e.

$$y = \frac{1}{4h} \frac{V_y}{V_0} z^2,$$

which is a parabolic trajectory.

Considering that the electron is transversely accelerated until $z = d$, the total angular deflection $\theta$ will be

$$\tan \theta = \left( \frac{\partial y}{\partial z} \right)_{z=d} = \frac{1}{2} \frac{d}{h} \frac{V_y}{V_0}.$$

and displacement $Y$ on the screen is

$$Y(V_y) = y(z = d) + \tan \theta D,$$

i.e.,

$$Y(V_y) = \frac{1}{2} \frac{d}{h} \frac{1}{V_0} \left( \frac{d}{2} + D \right) V_y.$$

$Y$ is indeed proportional to the voltage applied to the plates through a rather complicated proportional factor.

The geometrical and electrical parameters of this proportional factor play a fundamental role in the resolution of the instrument. In fact, the smaller the distance $h$ between the plates, or the smaller the gun voltage drop $V_0$, the larger is the displacement $Y$. Moreover, $Y$ increases quadratically with the electron beam distance $d$.

### C.4.1 CRT Frequency Limit

The electron transit time through the plates determine the maximum frequency that a CRT can plot. In fact, if the transit time $\tau$ is much smaller
than the period $T$ of the wave form $V(t)$, we have

$$V(t) \simeq \text{constant, \quad if \quad } \tau \ll T,$$

and the signal is not distorted.

The transit time is

$$\tau = \frac{d}{v_0} = d \sqrt{\frac{\mu}{2eV_0}}.$$

Supposing that

$$\begin{cases} V_0 = 1\text{kV} \\ d = 20\text{mm} \\ \mu c^2 \simeq 0.5\text{MeV} \\ e = 1\text{eV} \end{cases} \implies \tau \simeq 1\text{ns}$$
Appendix D

Perturbation of an Electronic Instrument

Every time we make a measurement, we always have to perturb the system we want to measure. This perturbation will produce systematic and random errors that can potentially compromise the measurement accuracy and precision. Here, we will consider only the case of voltage and current measurements, and in particular we will discuss the case of constant current and voltage measurements made on a generic electronic circuit and evaluate the systematic error introduced by the instrument. Then, we will extend the results to the more general case of AC circuits.

D.1 Current Measurement

Let’s consider a generic circuit with two terminals A and B represented on the left side of Figure D.1, and let’s suppose we need to measure the DC current flowing through A and B when the leads are connected. To determine the perturbation introduced by the instrument, we will use the Thevenin equivalent representation as shown on the right of Figure D.1.

Considering the Thenvenin equivalent circuit, the current flowing without and with the instrument is
APPENDIX D. PERTURBATION OF AN ELECTRONIC INSTRUMENT

Figure D.1: General circuit (left), and equivalent Thevenin circuit (right) with the instrument (amperometer).

\[ I = \frac{V}{R}, \]
\[ I' = \frac{V}{R + r}, \]

and therefore

\[ I' = \frac{1}{1 + \frac{r}{R}} I. \]

We can conclude that the smaller the ratio \( r/R \) the more accurate the measurement will be, and therefore a good amperometer should have an impedance as small as possible.

The currents difference is

\[ \Delta I = I - I' = \frac{r}{R + r} V. \]

Writing the relative systematic error, we will finally obtain

\[ \frac{\Delta I}{I} = \frac{r}{R + r} = \frac{r}{R} \frac{1}{1 + \frac{r}{R}} \approx \frac{r}{R} \left( 1 - \frac{r}{R} \right) \approx \frac{r}{R}. \]

In words, the relative systematic current perturbation to the circuit goes with the ratio of the instrument resistance and the resistance seen from the terminals A and B.
When we measure a current flowing through a branch of a circuit, we have to open the branch and place the instrument in series as we just did to measure $I$. Considering the generality of the Thevenin circuit equivalence, we can therefore reduce any circuit to the configuration shown in Figure D.1 and perform the current measurement as just described.

D.2 Voltage Measurement

Let’s suppose now that we want to measure the voltage across the two terminals A and B of the left circuit sketched in Figure D.2 and, again, we want to calculate the perturbation introduced by the instrument.

Using the voltage divider equation we have

$$V = \frac{R'}{R + R'}V_s$$

and therefore

$$V = \frac{1}{1 + \frac{R}{R'}}V_s.$$
We can conclude that the smaller the ratio $R / R'$ the more accurate the measurement will be. A good voltmeter should have an input impedance as large as possible.

Computing the relative uncertainty

$$\Delta V = V_s - V = \frac{R}{R + R'} V_s$$

$$\frac{\Delta V}{V_s} = \frac{R}{R + R'} = \frac{R}{R'} \frac{1}{1 + \frac{R}{R'}} \simeq \frac{R}{R'} \left(1 - \frac{R}{R'}\right) \simeq \frac{R}{R'}$$

In words, the relative voltage perturbation to the circuit goes with the ratio of the resistance seen from the terminals A and B and of the instrument resistance.

The same consideration made for current measurements holds for voltage measurements. We can always reduce any circuit to the configuration shown in Figure D.2 and perform the voltage measurement as just described.

**D.3 AC Circuit Measurement Perturbation**

It is straightforward to extend the previous results to the more general case of circuits in the AC regime. In fact, it suffices to replace resistances with the magnitude of a generic impedance.

For current measurements, we will have

$$\frac{|z|}{|Z|} \ll 1 \quad \Rightarrow \quad \frac{|\Delta I|}{|I|} = \frac{|z|}{|Z + z|} \simeq \frac{|z|}{|Z|}.$$  

For voltage measurements, we will have

$$\frac{|Z|}{|Z'|} \ll 1 \quad \Rightarrow \quad \frac{|\Delta V|}{|V|} = \frac{|Z|}{|Z' + Z|} \simeq \frac{|Z|}{|Z'|}.$$  

**D.3.1 Example: Oscilloscope**

For the oscilloscope, we will have

$$Z' = R_s + \frac{1}{j\omega C_s}.$$
Supposing that

\[ Z = R + \frac{1}{j\omega C}, \]

we will have

\[
\left| \frac{\Delta V}{V} \right| \approx \left| \frac{R + \frac{1}{j\omega C}}{R_s + \frac{1}{j\omega C_s}} \right|
\]

and therefore if

\[ R \ll R_s \]
\[ C \gg C_s \]

the oscilloscope voltage measurement will have a small systematic error.
Appendix E

Data Acquisition System
“Experimenter”

The “Experimenter” device is a very simple data acquisition system (DAQ) which allows to acquire several channels with a low data rate. This are its main characteristics:

• maximum sampling frequency $\nu_s = 100 \text{samples/s}$
• Timer resolution $\Delta T = 2 \text{ms}$

• **Input channels Characteristics**
  - Number : 4
  - Number of bits: $n = 10$
  - range : unipolar from 0V to 5.115V $\Rightarrow \Delta V_i = 5.115V$
  - resolution $\delta V_i = \Delta V_i / 2^n = 5 \text{mV}$

• **Output Channels Characteristics**
  - Number: 2
  - Number of bits: $n = 8$
  - type : unipolar
  - range from 0 to 5V, $\Rightarrow \Delta V_o = 5V$
  - resolution: $\delta V_o = \Delta V_o / 2^n \simeq 20 \text{mV}$ (not linear)
  - maximum current output: 1mA
The output channel control is implemented using the so called pulse with modulation (PWM) technique. Essentially, a PWM voltage source works by changing the time that the pulse is on (the duty cycle), and sending the pulse to a low pass filter.

The non linearity of the output channel is clearly shown in the plot shown in figure E.1. The reason of such non-linearity comes from the way that PWM is implemented.
E.2 ExperTerm Program

The “ExperTerm” is a terminal interface to communicate via RS232 port to the DAQ. This is the list of the available commands:

- **a ch0 [ch1] ...** print ADC sampled value of channel ch0 ch1 ... Example: a 1
- **A ch0 [ch1] ...** continuously print ADC sampled values of channel ch0 ch1 ... until a ctrl c is pressed. Example: A 1 0 3
- **e ch val** set voltage source “ch” to “val”. Example: e 0 124
- **h** print this help screen
- **q** quit program

E.3 ExperDAQ Program

The “ExperDAQ” is a command line program for continuous data acquisition with the Experimenter. It allows to specify the acquisition parameters (input channels, number of samples, sampling frequency ) to change internal relay status, and to set one voltage source. The following text is the program usage help:

```
usage: ExperDAQ [-v] [-p Port,BaudRate,Parity,Bits,StopBit] [-o Channel,A,B,N] [-r] ChannelList Samples SamplingRate Averages FileName
```

Parameters between square brackets are optional.

- **-v** verbose mode.
- **-p** serial port configuration example: -p 1,9600,0,8,1
- **-o Channel,A,B,N** source output settings. Trailing spaces are not allowed
  - **Channel**: 0 or 1
  - **A**: initial value from 0 to 255
  - **B**: final value from 0 to 255
  - **N**: number of step to go from A to B. The step is done every 1/SampleFrequency
• -r change relay status before data acquisition and restore previous status after the acquisition

• **ChannelList**: list of channels to acquire from 0 to 7. Values are separated with a comma. Trailing spaces are not allowed. Example: 3,0,1

• **Samples number**: samples to acquire

• **SamplingRate**: samples per seconds to acquire from 1 to 20

• **Averages**: number of averages. 0 for no averages. The data used to compute the average are collected at the SamplingRate

• **FileName**: filename containing the acquired data. First column is the time. The other columns are the channels values specified in the ChannelList

**Example:**

`ExperDAQ -r -o0,0,255,20 0,1,2 20 10 0 data.txt`

The previous example does the following: change relay status during the acquisition, sweep the voltage source 0 from 0 to 255, acquires 20 samples of ADC channels 0,1,2 with a sampling rate of 10sample/s, does not perform averages, and finally saves the data into the file data.txt.