An Introduction to Spinors

by Eric Thrane

University of Washington

March 27, 2007

In this article we're going to study the properties of spinors. Understanding spinors will give us tools to investigate many interesting properties of neutrinos.

The Lorentz group is the group of rotations and boosts in 3+1 dimensions. There are three generators for rotations and three generators for boosts for a total of six generators. As always, group elements are obtained by exponentiating the generators of the group:

$$L_{(\chi)} = e^{-i\Lambda_z \chi}$$

$L_{(\chi)}$ is an element of the Lorentz group and Λ_z is a generator for the Lorentz group. Topology aside[1], the structure of the group is entirely encoded in the algebra of the group, which is just the set of commutation relations between generators. The algebra of SU(2), for instance, is $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}$. The algebra of the Lorentz group is[2]

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\left(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}\right) \tag{1}$$

[1]See the appendix for a discussion on the fascinating topic of group topology.

[2]You might rightly ask where this algebra comes from. Consider the generators for rotations in SO(3): $\mathbf{J} = \mathbf{x} \times \mathbf{p}$. We can use the commutation relation, $[x_i, p_j] = i\delta_{ij}$, to show that the algebra for the $\{J_i\}$ is given by $J_i = i\epsilon_{ijk}[J_j, J_k]$...the same algebra as SU(2)! So that's SO(3)...to extend this analysis to SO(3,1), we simply promote the three-vectors to Lorentz four vectors. There is no 4D analogue of a cross product, however, so we need to rewrite our definition of J_i so it can be extended easily to 3+1 dimensions. To do this we organize the generators into an antisymmetric matrix as in equation (2): $J_{ij} = x_i p_j - x_j p_i$, where $J_i = \epsilon_{ijk} J_{jk}$. Then, when we add a dimension, $J_i j$ becomes $J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu$. The Lorentz version of the canonical commutation relation is now: $[x_\mu, p_\nu] = i g_{\mu\nu}$. Armed with this alternate definition of $J_{\mu\nu}$ and the canonical commutation relation, the interested reader can now derive equation (1).

where $J^{\mu\nu}$ is an antisymmetric tensor defined as:

$$J^{\mu\nu} = \begin{pmatrix} 0 & \Lambda_x & \Lambda_y & \Lambda_z \\ -\Lambda_x & 0 & R_z & -R_y \\ -\Lambda_y & -R_z & 0 & R_x \\ -\Lambda_z & R_y & -R_x & 0 \end{pmatrix}. \tag{2}$$

$\{\Lambda_i\}$ are boosts and $\{R_i\}$ are rotations.

Now it's time to discuss representations of the Lorentz group. A representation is a set of matrices that satisfy the group multiplication rules. Each group usually has lots of different representations. Consider, for example, rotation matrices in quantum mechanics. A particle lives in a representation determined by its spin. The generator J_z is different depending on the total spin, $\mathbf{J^2}$, of the particle:

$$J_z^{1/2} = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \qquad J_z^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

In this case the two representations have different dimensions, but it is also possible to have two or more representations with the same dimensionality. In order for two representations to be counted as different, however, they can not be related by a similarity transformation: $G' = SGS^{-1}$. There is usually physical significance attached to the representation an object lives in.

When most people think of the Lorentz group, they first visualize the "defining representation," which is the representations where Lorentz four-vectors live. E.g.:

$$\Lambda_z = \begin{pmatrix} cosh(\chi) & 0 & 0 & -sinh(\chi) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -sinh(\chi) & 0 & 0 & cosh(\chi) \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix}. \tag{3}$$

χ is rapidity. As noted above, Lorentz four-vectors live in this representation. So do electromagnetic fields:

$$F_{\mu\nu} \rightarrow L_\mu^\alpha L_\nu^\beta F_{\alpha\beta}.$$

Here $L_\mu^\alpha, L_\nu^\beta$ are in the defining representation. In fact, just about anything with a greek index on it transforms in the defining representation. Spin-1/2 particles, however, do not live in the defining represenation.

Here's why. Rotations are a subgroup of the Lorentz group, and we already know how spin-1/2 particles transform under rotations: in the spin-1/2 representation of SU(2):

$$R^z_{(\theta)} = e^{-\frac{i}{2}\theta\sigma_z}$$

Rotations in the defining representation, however, look like SO(3) rotations:

$$R^z_{(\theta)} = \begin{pmatrix} cos(\theta) & 0 & -sin(\theta) \\ 0 & 0 & 0 \\ sin(\theta) & 0 & cos(\theta) \end{pmatrix}.$$

So we need a different representation that has SU(2) rotations built into it.

Now that we know what we want, we can build the representation we're looking for. We know that the SU(2) algebra is buried inside the Lorentz algebra. Can we find a set of generators, (or linear combinations of them), that satisfy the SU(2) algebra? We sure can if we pick the following linear combingations of generators:

$$\mathbf{A} = \frac{1}{2}(\mathbf{L} + i\mathbf{K})$$

The reader can verify that $[A_i, A_j] = \epsilon_{ijk} A_k$. Great, there's an SU(2) subalgebra! What about the remaining three generators? If we define:

$$\mathbf{B} = \frac{1}{2}(\mathbf{L} - i\mathbf{K})$$

we can show that the $\{B_i\}$ also form an SU(2) subgroup. Additionally, it is easy to verify that $[A, B] = 0$, which means that the Lorentz algebra decomposes into two separate SU(2) alegbras[3]. It's now clear that different representations of the Lorentz group correspond to pairs of representations of the two SU(2) sub-algebras! We can label a representation of the Lorentz group with a pair of half-integers: (j_1, j_2).

The simplest representation of the Lorentz group is $(0,0)$, which corresponds to a Lorentz scalar. In this representation, called the "trivial representation," each group element is represented by 1 (and each alegbra element is represented by 0). The second simplest representation is $(\frac{1}{2}, 0)$ or $(0, \frac{1}{2})$.

[3]This does not mean, however, that the Lorentz group is isomorphic to SU(2)$\oplus$SU(2). We must remember topology as well.

Here are the generators for $(\frac{1}{2},0)$: $\{A_i\} = \{\sigma_i\}$ and $\{B\} = 0$ Solving for the physically significant generators, $\{\Lambda_i\}$ and $\{L_i\}$, we obtain: $\{L_i\} = \{\frac{\sigma_i}{2}\}$ and $\{\Lambda_i\} = \{\frac{-i\sigma_i}{2}\}$. If we repeat this exercise for $(0,\frac{1}{2})$, we find $\{A\} = 0$ and $\{B_i\} = \{\sigma_i\}$ and thus: $\{L_i\} = \{\frac{\sigma_i}{2}\}$ and $\{\Lambda_i\} = \{\frac{+i\sigma_i}{2}\}$.

Did you notice that these two representations have identical representations for $\{L_i\}$, but that the $\{\Lambda_i\}$ differ by a minus sign? It turns out that the representation $(\frac{1}{2},0)$ is the appropriate representation for a left-handed[4] spinor, which is a two-component object made out of a left-handed particle field[5] and a right-handed anti-particle field. E.g.:

$$\psi_L = \begin{pmatrix} e_L^- \\ e_R^+ \end{pmatrix}$$

Here e_L^- is meant to be a left-handed electron field and e_R^+ is meant to be a right-handed positron field. The $(0,\frac{1}{2})$ representation, on the other hand, corresponds to a right-handed spinor, which is made of a right-handed particle and a left-handed anti-particle. E.g.:

$$\psi_R = \begin{pmatrix} e_R^- \\ e_L^+ \end{pmatrix}$$

Since $(0,\frac{1}{2})$ and $(\frac{1}{2},0)$ have the same generators for rotations, ψ_L and ψ_R transform the same way under rotations. This means that rotations are mathematically identical for particles and antiparticles, which is intuitive. The generators for boosts, however, differ by a minus sign for ψ_L and ψ_R, so boosts are another matter.

The two component spinors, ψ_L and ψ_R, are sometimes called Majorana spinors (and also Weyl spinors) because they can be used to build the Lagrangian for a Majorana particle, where there is no distintion between a particle and an antiparticle. No particle in the standard model can be a Majorana particle except the neutrino, because all other particles are charged, and so there is a clear distinction between particle and antiparticle. Since there is a distinction between charged particles and their anti-particles, it is necessary to have four degrees of freedom to describe a charged particle's

[4]The term "left-handed" refers to something known as the "chirality" of a particle. It is closely related to a measurable quantity "helicity," which is the spin in the direction of propagation. We'll discuss chirality in more detail soon.

[5]Particle fields, if you are not already familiar with them, are kind of like multi-particle wavefunctions. Take a look in your favorite QFT book for a formal definition.

handedness and charge at the same time. This is easily accommodated by taking the tensor sum of two Majorana spinors:

$$\psi_D = \psi_L \oplus \psi_R = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} e_L^- \\ e_R^+ \\ e_R^- \\ e_L^+ \end{pmatrix}$$

The resultant four-component spinor is called a "Dirac spinor," and it lives in the representation: $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$.

At this point it's time to introduce the famous Dirac matrices to facilitate further discussion of Dirac spinors. We'll be using the chiral basis (also called the Weyl basis), but there are other bases in use, which are related by similarity transformations.

$$\gamma_0 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma_i \equiv \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

These are four-by-four matrices, so the 1 should be interpreted as a two-by-two identity matrix. The gamma matrices obey a very convenient anti-commutation relation, $\{\gamma_\mu, \gamma_\nu\} = ig_{\mu\nu}$, and they can be used to write the Dirac representation in a compact form: $J_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu]$. They come up again and again in Lagrangians and equations of motion for spin-1/2 particles where they always act on Dirac spinors. Another nice feature about the gamma matrices $\{\gamma_\mu\}$ is that they have a Greek index on them which is taken literally—they transform like Lorentz four-vectors. Just to reiterate, as this can be confusing, the $\{\gamma_\mu\}$ are four matrices that act on Dirac spinors, but themselves transform in the defining representation like a traditional Lorentz four-vector.

Speaking of the defining representation, how does it fit into the (j_1, j_2) scheme? By dimensional analysis, we know it must be $(0, 1)$, $(1, 0)$ or $(\frac{1}{2}, \frac{1}{2})$. The answer is that it corresponds to the representation $(\frac{1}{2}, \frac{1}{2})$. It's interesting to prove this on your own. See [1], problem 3.1 for guidance.

So why did I choose to make spinors the topic of the second lecture in this series on neutrino physics? What are we going to do with them? Let's take a look ahead. The Dirac spinor, ψ, and it's cousin, $\bar{\psi} \equiv \psi^\dagger \gamma_0$, appear in the basic Lagrangian we'll be studying:

$$\mathcal{L} = \bar{\psi}(i\partial_\mu \gamma^\mu - m)\psi$$

The classical equation of motion for this Lagrangian is the Dirac equation:

$$(i\partial_\mu \gamma^\mu - m)\psi_{(x^\mu)} = 0$$

(You already have the tools to show that this Lagrangian and EOM are invariant under Lorentz transformations...why don't you give it a try?) When we discuss Majorana neutrinos we'll use the two-component Weyl spinors discussed above and we'll be able to show that they violate lepton number. Spinors will also play a roll in our discussion of neutrino CP violation. They'll be essential to our discussion of the weak force too, which only talks to left-handed particles:

$$\mathcal{H} = \frac{G_F}{\sqrt{2}} \left[\bar{e}_L \nu_{eL}\right] \left[\bar{\nu}_{eL} e_L\right] + ...$$

Handedness in general is a very important concept for neutrinos; no one has ever seen a right-handed neutrino!

Appendix

You can associate geometric shapes with groups. The group U(1), for instance, can be associated with a circle. Each group element can be parameterized by a single angle on $[0, 2\pi)$: $G = e^{i\theta}$. Thus we say that U(1) is topologically equivalent to a circle, or 1-sphere. SU(2), it turns out, is topologically equivalent to a 3-sphere. To see this, we first note that any element of SU(2) can be parameterized as follows:

$$U = \begin{pmatrix} \alpha & \beta^* \\ -\beta & \alpha^* \end{pmatrix} = \begin{pmatrix} x+iy & u-iv \\ -u-iv & x-iy \end{pmatrix}$$

where $\alpha \equiv x + iy$ and $\beta \equiv u + iv$ are complex numbers subject to the constraint that $|\alpha|^2 + |\beta|^2 = 1$. We can write such a matrix as a linear combination of pauli matrices and the identity:

$$U = x\mathbb{I} + iu\sigma_x + iy\sigma_z - iv\sigma_y \tag{4}$$

subject to the constraint that $|\alpha|^2 + |\beta|^2 = x^2 + y^2 + u^2 + v^2 = 1$. If we take $\{\mathbb{I}, \sigma_x, \sigma_y, \sigma_z\}$ as unit vectors in a four-dimensional space, it is apparent that equation (4) describes a three-sphere.

Now consider SO(3). Any rotation in SO(3) can be described with two parameters: a unit vector $\hat{n}$, about which the rotation occurs and an angle, γ, that describes the size of the rotation. The vector $\hat{n}$ can be in any direction, so we can parameterize it with the usual spherical coordinates: (θ, ϕ). The angle γ runs from $[0, \pi]$. (If you want to rotate past π, just flip $\hat{n} \to -\hat{n}$ and you can accomplish the desired rotation with γ on $[0, \pi]$.) Any SO(3) rotation can be parameterized by $\{\theta, \phi, \gamma\}$, but there is a degeneracy: $R_{(\hat{n},\gamma=\pi)} = R_{(-\hat{n},\gamma=\pi)}$. Without the degeneracy, each group element could be described by one azimuthal angle (on $[0, 2\pi]$), and two polar angles (on $[0, \pi]$). One azimuthal angle and two polar angles also describe the surface of 3-sphere. Because of the degeneracy, however, the point $P : (\theta, \phi, \gamma)$ is equivalent to the point $Q : (\pi - \theta, -\phi, \gamma)$, which is on the opposite side of the 3-sphere. This means every point on the 3-sphere is identified with a point opposite the center of the 3-sphere. (Just picture a clock where 12:00 is the same time as 6:00.) This is called "antipodal identification."

Since SU(2) and SO(3) have the same algebra, they have similar properties for infinitessimal rotations. The fact they have different topologies implies that they have different global structure. The best way to illustrate this is to consider a rotation about the z-axis by 2π. In SO(3), $G_{(2\pi,\hat{z})} = 1$, but in SU(2), $G_{(2\pi,\hat{z})} = -1$. In SU(2), you have to go around 4π to get back to one.

References

[1] Peskin and Schroeder *An Introduction to Quantum Field Theory* 1995