Physics 106a, Classical Mechanics
Mid-term Exam, Fall Term, 2019

Do not read beyond this page until you are ready to start the exam!

Material: The exam covers the material of Lectures 1-7, up to "Driven, Damped Oscillations".

Due date: Due in Ph106 box, East Bridge, on Friday 7pm, November 1, 2019. Late exams will require extenuating circumstances; otherwise, no credit will be given. The "one free extension" *cannot* be used on the mid-term.

Logistics: The exam consists of four problems, on pages 2 to 5 of this file. Do not look at the problems until you are ready to start it. Please use a blue book for the exam. Typed exams on individual pages are allowed, but all the typing must be completed in the allotted time. There are **four** questions.

Time: Four hours maximum. You may take breaks, not counted in the 4 hours; but please limit the time spent on breaks. Of course, you should not be consulting references or working on the problems during the breaks. You are under the honor code!

Resources: The midterm and final are not collaborative. All questions must be done on your own, without consulting anyone else. You may consult your own notes (both in-class and notes on this class you or a colleague in the class have made), the text by Hand and Finch or by Taylor, and lecture notes and solution sets on the Ph 106a website. You may not consult any other material, including other textbooks, the web (except for the current Ph106a website), material from previous years' Ph106 or any other classes, or copies you have made of such material, or any other resources. Calculators and symbolic manipulation programs are not allowed or needed.
If these instruction are not clear, please consult with me before starting the exam.

Grading: Points for each part of each problem are noted in square brackets, e.g. [2 points]. The exam counts for approximately 15% of the total term grade. Midterm grades reported to the registrar will be P/F. You will get a P if you handed in A1, A2, A3 and this midterm; else, F.

Reminder: Due date: Friday 7pm, 1 November, 2019

Problem 1 - Miscellaneous questions

This problem consists of 5 independent short problems.

(a) **Spring pendulum**

[5 points] A pendulum is made from a mass m hanging in gravity g on a spring with the other end fixed. The undistorted length of the spring is l and the spring constant is k. Find the Lagrangian for the full three dimensional motion using spherical polar coordinates r, θ, ϕ with the origin at the fixed end of the spring. What are the constants of the motion, if any?
Solution:

The kinetic energy is

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\dot{\phi}^2). \tag{1}$$

The potential energy is

$$V = \frac{1}{2}k(r-l)^2 + mgr\cos\theta \tag{2}$$

The Lagrangian is

$$\mathcal{L} = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2) - \frac{1}{2}k(r-l)^2 - mgr\cos\theta \tag{3}$$

Notice that the coordinate ϕ is not present explicitly in the Lagrangian, and hence it is *ignorable / cyclic*. Therefore, the conjugate momentum

$$p_\phi = \partial L/\partial\dot{\phi} = mr^2\sin^2\theta \tag{4}$$

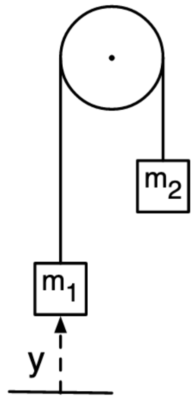
is a constant of the motion.

The Lagrangian does not depends explicitly on time, and so the Hamiltonian H is constant. The kinetic energy is a quadratic form in the time derivatives of the coordinates, which means $H = T + V$.

- **(1 pt)** : Correct expression for the kinetic energy.
- **(1 pt)** : Correct expression for the potential energy.
- **(1 pt)** : Identifying that the conjugate momentum p_ϕ is a constant of motion.
- **(1 pt)** : Identifying that the Hamiltonian is a constant of motion.
- **(1 pt)** : Identifying that the total energy (which is the Hamiltonian in this case) is a constant of motion.

Total sub-points : 5

(b) **Pulley with friction**



[**4 points**] Two masses m_1, m_2 are connected by a massless rope running over a pulley with moment of inertia I and radius a rotating about a fixed axle. There is no slip between the rope and the pulley, so that the pulley rotates when the masses move. However due to friction in the axle, the pulley requires a torque (force times moment arm) $\eta\omega$ with ω the rotation rate. Find the virtual work for a virtual displacement δy when the mass m_1 is at position y and moving upwards with speed $\dot{y}$. What is the generalized force $\mathcal{F}_y$?

Solution:

For a virtual displacement δy, the mass m_1 moves up by δy, and the mass m_2 moves down by δy, giving a net virtual work $(m_2 - m_1)g\delta y$.

Also, each point on the rope around the pulley moves δy tangentially to the pulley, giving an additional contribution to the virtual work $-F_f\delta y$, with F_f the total force giving the torque needed to make the pulley rotate against the axle friction, i.e.

$$F_f a = \eta\omega = \frac{\eta\dot{y}}{a}. \tag{5}$$

Note that the force on the rope is in the direction opposite to the motion.

Thus the total virtual work is

$$\Delta W = \left[(m_2 - m_1)g - \frac{\eta\dot{y}}{a^2}\right]\delta y \tag{6}$$

and the generalized force is

$$\mathcal{F}_y = \left[(m_2 - m_1)g - \frac{\eta\dot{y}}{a^2}\right]. \tag{7}$$

Remember that the virtual work is calculated for a small displacement at constant time, and hence the velocities fixed.

- **(1 pt)** : Correct virtual work contribution by gravitational force.
- **(1 pt)** : Correct expression for the total force F_f to rotate the pulley against axle friction.
- **(1 pt)** : Correct virtual work by anti-axle friction force.
- **(1 pt)** : Correct expression for the generalized force.

Total sub-points : 4

(c) **Bead on wire:**

[**4 points**] A bead of mass m slides in gravity g on a frictionless wire in a vertical plane with shape described by the equation $x^3 + xz + z^3 = 1$ (with x horizontal, z vertical). Find the complete set of equations that could be analyzed (e.g. put on a computer) to give the motion.

Solution:

The Lagrangian in this case, in terms of the generalized coordinates (x, z), is simply

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{z}^2) - mgz \tag{8}$$

The constraint can be included using the Lagrange multiplier method. With constraints, the Lagrange's equation can be written as

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right) - \frac{\partial \mathcal{L}}{\partial q_k} + \lambda_\alpha \frac{\partial f_\alpha}{\partial q_k} = 0 \tag{9}$$

The constraint is

$$f = x^3 + xz + z^3 - 1 = 0 \tag{10}$$

With the Euler-Lagrange equations, we have

$$\begin{cases} m\ddot{x} - \lambda(3x^2 + z) = 0, \\ m\ddot{z} + mg - \lambda(3z^2 + x) = 0. \end{cases} \tag{11}$$

Together with constraint, these three equations can be analyzed (using a computer) to give the motion.

- **(1 pt)** : Correct expression for the Lagrangian.
- **(1 pt)** : Correct expression for the constraint.
- **(1 pt)** : Getting the correct equation of motion for x using the Lagrange's equation.
- **(1 pt)** : Getting the correct equation of motion for y using the Lagrange's equation.

Total sub-points : 4

(d) **The Hamiltonian:**

[3 points] The kinetic and potential energies of a system with a single degree of freedom characterized by the generalized coordinate q are

$$T = \frac{1}{2}m\dot{q}^2 + \frac{1}{2}m\omega^2 \sin^2\alpha\, q^2, \quad V = mg\cos\alpha\, q, \tag{12}$$

with m, g, α, ω constants. What is the Hamiltonian as a function of coordinate and momentum (not velocity)?

Solution:

The Lagrangian is $L = T - V$ and the momentum is $p = \partial L/\partial \dot{q} = m\dot{q}$. The Hamiltonian is

$$H = p\dot{q} - L = \tfrac{1}{2}m\dot{q}^2 - \tfrac{1}{2}m\omega^2\sin^2\alpha\, q^2 + mg\cos\alpha\, q, \tag{13}$$

$$= \tfrac{1}{2m}p^2 - \tfrac{1}{2}m\omega^2\sin^2\alpha\, q^2 + mg\cos\alpha\, q. \tag{14}$$

Note that this is an example where $H \neq T + V$ because the kinetic energy is not a *quadratic form* in the velocity $\dot{q}$, and the $\frac{1}{2}m\omega^2\sin^2\alpha$ term comes in with *opposite* sign in L and H. You might recognize the Lagrangian as the one for a particle on a rotating wire. In this case the $\frac{1}{2}m\omega^2\sin^2\alpha$ term is the part of the kinetic energy coming from the angular motion.

- **(1 pt)** : Correct expression for momentum $p = m\dot{q}$.
- **(2 pt)** : Correct expression for Hamiltonian in terms of coordinate and momentum. If students only wrote down the Hamiltonian in terms of coordinate and velocity, award 1 point for their effort.

Total sub-points : 3

(e) **Driven oscillator:**

[5 points] A simple harmonic oscillator is described by the equation of motion

$$\ddot{q} + q = F(t), \tag{15}$$

with $F(t)$ being the driving force, and is initially at rest at $q = 0$. Find the solution for $q(t)$ for $t > 0$ if an oscillating force $F(t) = \sin(2t)$ is switched on at time $t = 0$.

Solution:

The general solution is the sum of the particular solution and complementary solution

$$q(t) = -\tfrac{1}{3}\sin 2t + A\cos t + B\sin t, \tag{16}$$

where the coefficient of the $\sin 2t$ term is given by substituting a trial solution $q = C\sin 2t$ (remember *any* solution will do for the particular integral).

The initial conditions $q(0) = \dot{q}(0) = 0$ gives $A = 0, B = \frac{2}{3}$, so the solution is

$$q(t) = \tfrac{2}{3}\sin t - \tfrac{1}{3}\sin 2t. \tag{17}$$

Comment: It's important to remember that the initial conditions are imposed on the full solution, not the complementary function. You could also use the Green function method, although the integrals are rather messy.

- **(2 pt)** : Correct particular solution with correct coefficient. (1 pt for solution, 1 pt for coefficient)
- **(1 pt)** : Using the initial condition to get $A = 0$
- **(1 pt)** : Using the initial condition to get $B = \frac{2}{3}$
- **(1 pt)** : Correct final expression for the solution.

Total sub-points : 5

Problem 2 - Simple harmonic oscillator with time-dependent Lagrangian

Assume the Lagrangian for a certain one-dimensional motion is given by

$$L = e^{\phi t}(\frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2) \tag{18}$$

where ϕ, m, and k are positive constants.

(a) **[2 points]** Write down the Euler-Lagrange equation. Then apply the equation to the Lagrangian and obtain the equation of motion.

Solution:

The Euler-Lagrange equation is:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0 \tag{19}$$

The equation of motion for q is then

$$e^{\phi t}(m\ddot{q} + \phi m\dot{q} + kq) = 0 \tag{20}$$

$$\ddot{q} + \phi\dot{q} + \frac{kq}{m} = 0 \tag{21}$$

- **(1 pt)** : Correct Euler-Lagrange equation.
- **(1 pt)** : Correct equation of motion. If students keep the $e^{\phi t}$ in the equation, the point should still be awarded.

Total sub-points : 2

(b) **[8 points]** Solve the equation by trying q in exponential form e^{At}, and discuss the different solutions in different conditions.

Solution:

We try solutions of the form $q \sim e^{At}$. Substitution gives:

$$A^2 + \phi A + \frac{k}{m} = 0 \tag{22}$$

with solution :

$$A = -\frac{\phi}{2} \pm \sqrt{\frac{\phi^2}{4} - \frac{k}{m}} \tag{23}$$

Write this as $A = -\frac{\phi}{2} \pm b$, and consider the three possible cases.

(a) $\frac{\phi}{2} < \sqrt{\frac{k}{m}}$, then b is imaginary; let it be $i\beta$. The general solution is:

$$q = e^{-\frac{\phi t}{2}}(C_1 \cos\beta t + C_2 \sin\beta t) \tag{24}$$

The motion is oscillatory with attenuating amplitude.

(b) $\frac{\phi}{2} = \sqrt{\frac{k}{m}}$, then $b = 0$ and we have:

$$q = q_0 e^{-\frac{\phi t}{2}} \tag{25}$$

showing that the motion is non-oscillatory with q attenuating from the value q_0 at $t = 0$.

(c) $\frac{\phi}{2} > \sqrt{\frac{k}{m}}$, $b > 0$ and

$$q = e^{-\frac{\phi t}{2}}(C_3 e^{bt} + C_4 e^{-bt}) \tag{26}$$

This motion is also non-oscillatory and time attenuating.

- **(1 pt)** : Substituting $q\ e^{At}$ into the equation, and simplifying the equation.
- **(1 pt)** : Correct solution for the trial function.
- **(2 pt)** : Correct solution and discussion for $\frac{\phi}{2} < \sqrt{\frac{k}{m}}$.
- **(2 pt)** : Correct solution and discussion for $\frac{\phi}{2} = \sqrt{\frac{k}{m}}$.
- **(2 pt)** : Correct solution and discussion for $\frac{\phi}{2} > \sqrt{\frac{k}{m}}$.

Total sub-points : 8

(c) **[8 points]** Suppose a point transformation is made to another generalized coordinate, given by $S = e^{\frac{\gamma t}{2}} q$. What is the Lagrangian in terms of S? Find Lagrange's equation, and possible constants of motion. Describe the relationship of new solutions with results of part (b).

Solution:

Applying the point transformation, we will have the Lagrangian to be:

$$L = e^{(\phi-\gamma)t}(\frac{1}{2}m(\dot{S} - \frac{\gamma}{2}S)^2 - \frac{1}{2}kS^2) \tag{27}$$

And the equation of motion is:

$$\ddot{S} + (\phi - \gamma)\dot{S} + (\frac{1}{4}\gamma^2 - \frac{\gamma\phi}{2} + \frac{k}{m})S = 0 \tag{28}$$

If $\phi = \gamma$:

With $\omega^2 = \frac{k}{m} - \frac{\gamma^2}{4}$. As $\ddot{S} = \frac{1}{2}\frac{d\dot{S}^2}{dS}$, integration gives:

$$\dot{S}^2 + \omega^2 S^2 = \text{constant.} \tag{29}$$

Hence there is a constant of motion.

For $\frac{\gamma}{2} < \sqrt{\frac{k}{m}}$, ω is real, and the equation of motion in S describes a simple harmonic motion without damping. For $\frac{\gamma}{2} = \sqrt{\frac{k}{m}}$, $\omega = 0$ and the motion in S is uniform. For $\frac{\gamma}{2} > \sqrt{\frac{k}{m}}$, ω is imaginary and the motion is non-oscillatory with time attenuation. And S contains the attenuating factor $e^{-\frac{\gamma t}{2}}$ which causes time attenuation in all the three cases.

If $\phi \neq \gamma$:

There's no constant of motion.

Let $S \sim e^{A_1 t}$, we have:

$$A_1^2 + A_1(\phi - \gamma) + (\frac{1}{4}\gamma^2 - \frac{\gamma\phi}{2} + \frac{k}{m}) = 0 \tag{30}$$

so $A_1 = \frac{\gamma-\phi}{2} \pm \sqrt{\frac{\phi^2}{4} - \frac{k}{m}}$, and the remained discussion is exactly the same as part b).

- **(1 pt)** : Correct expression for Lagrangian in terms of S.
- **(1 pt)** : Correct expression for Lagrange's equation.
- **(1 pt)** : Considering the case of $\phi = \gamma$, and find that there is a constant of motion.
- **(3 pt)** : Correct discussion on the solution for S under the three cases. 1 point is awarded for each case discussed.
- **(1 pt)** : Considering the case of $\phi \neq \gamma$, and find that there are no constants of motion.
- **(1 pt)** : Solving the equation of motion with a trial function similar to that in part (b).

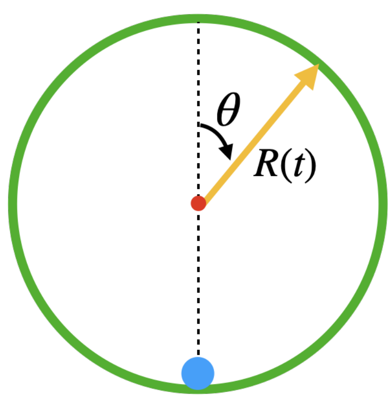
Total sub-points : 8

Problem 3 - Particle moving on a circular track with time-varying radius

A point particle of mass m is set to move on a vertical circular track at all time internally (see the figure below) of radius $R(t)$. This is a strange circular track because its radius changes with time, in particular the radius is of the form

$$R(t) = R_0\left(1 + \frac{a}{2}\sin t\right), \tag{31}$$

where R_0 and a are unspecified constants such that $R > 0$ for all t. Initially the particle at the bottom of the track. It is then given a kick at time $t = 0$ such that it has a certain initial angular velocity ω_0. In this problem, use polar coordinates (r, θ) to describe the position of the particle. Take the acceleration due to gravity (which is acting downward) as g.



(a) **[2 points]** Write down the Lagrangian for the particle in terms of the generalized coordinates (r, θ). Do not impose any constraints to the Lagrangian yet.

Solution:

The Lagrangian is simply given by

$$\mathcal{L} = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - mgr\cos\theta \tag{32}$$

- **(1 pt)** : Correct expression for the kinetic energy.
- **(1 pt)** : Correct expression for the potential energy.

Total sub-points : 2

(b) **[3 points]** What is the constraint in this problem? Choose the right word(s) from {*holonomic, non-holonomic, rheonomous (depends explicitly on time)* and *scleronomous (does not depend explicitly on time*} to describe the constraint.

Solution:

The constraint in the problem is

$$f = R - r = R_0\left(1 + \frac{a}{2}\sin t\right) - r = 0 \tag{33}$$

This constraint is **holonomic** (since it only involves the generalized coordinates and time) but **rheonomous** (it depends explicitly on time).

- **(1 pt)** : Correct constraint.
- **(1 pt)** : Stating that the constraint is holonomic.
- **(1 pt)** : Stating that the constraint is rheonomous.

Total sub-points : 3

(c) **[3 points]** Apply the Euler-Lagrange equation with Lagrange multipliers to find the equation of motion for r and θ.

Solution:

The Lagrange's equation with constraint is given by

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right) - \frac{\partial \mathcal{L}}{\partial q_k} + \lambda_\alpha \frac{\partial f_\alpha}{\partial q_k} = 0 \tag{34}$$

The equation of motion for r is then,

$$\frac{d}{dt}(m\dot{r}) - (mr\dot{\theta}^2 - mg\cos\theta) + \lambda\frac{\partial f}{\partial r} = 0 \tag{35}$$

$$m\ddot{r} - mr\dot{\theta}^2 + mg\cos\theta - \lambda = 0 \tag{36}$$

and the equation of motion for θ is then

$$\frac{d}{dt}(mr^2\dot{\theta}) - (mgr\sin\theta) + \lambda\frac{\partial f}{\partial \theta} = 0 \tag{37}$$

$$2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} - mgr\sin\theta = 0 \tag{38}$$

- **(2 pt)** : Correct equation of motion for r with Lagrange multiplier.

- **(1 pt)** : Correct equation of motion for θ

Total sub-points : 3

(d) **[5 points]** Using your constraint in (b), further reduce your equations in (c) to a pair of equations for θ and λ only (i.e. eliminate r from your equations). You are *NOT* required to solve the resulting equations.

Solution:

Since we have the constraint

$$r = R_0\left(1 + \frac{a}{2}\sin t\right) \tag{39}$$

We have

$$\begin{cases} \dot{r} = \frac{aR_0}{2}\cos t \\ \ddot{r} = -\frac{aR_0}{2}\sin t \end{cases} \tag{40}$$

The equation of motion for r can then be written as,

$$m\ddot{r} - mr\dot{\theta}^2 + mg\cos\theta - \lambda = 0 \tag{41}$$

$$m\left(-\frac{aR_0}{2}\sin t\right) - m\left[R_0\left(1 + \frac{a}{2}\sin t\right)\right]\dot{\theta}^2 + mg\cos\theta - \lambda = 0 \tag{42}$$

and the equation of motion for θ can then be written as

$$2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} - mgr\sin\theta = 0 \tag{43}$$

$$2m\left[R_0\left(1 + \frac{a}{2}\sin t\right)\right]\left(\frac{aR_0}{2}\cos t\right)\dot{\theta} + m\left[R_0\left(1 + \frac{a}{2}\sin t\right)\right]^2\ddot{\theta} - mg\left[R_0\left(1 + \frac{a}{2}\sin t\right)\right]\sin\theta = 0 \tag{44}$$

$$2m\left(\frac{aR_0}{2}\cos t\right)\dot{\theta} + m\left[R_0\left(1 + \frac{a}{2}\sin t\right)\right]\ddot{\theta} - mg\sin\theta = 0 \tag{45}$$

In principle, we have a pair (indeed, the two equations are the same. Check it by differentiating the first one with respect to time) of 2^{nd} order differential equations for θ and hence we should be able to solve for θ.

- **(3 pt)** : Expressing r, $\dot{r}$ and $\ddot{r}$ in terms of t.
- **(1 pt)** : Substituting r, $\dot{r}$ and $\ddot{r}$ into the equation of motion for r.
- **(1 pt)** : Substituting r, $\dot{r}$ and $\ddot{r}$ into the equation of motion for θ.

Total sub-points : 5

(e) **[5 points]** Let's turn the circular track back to a normal track, i.e. let's set $a = 0$. Find the solution for $\dot{\theta}$ in terms of θ *starting from the Euler-Lagrange equation.* (Hint: Consider the 2^{nd} equation. Find an integrating factor that makes the equation into an exact differential.)

Solution:

When $a = 0$, the equation of motions becomes

$$\begin{cases} -mR_0\dot{\theta}^2 + mg\cos\theta - \lambda = 0 \\ mR_0\ddot{\theta} - mg\sin\theta = 0 \end{cases} \tag{46}$$

Consider the 2^{nd} equation, multiplying both sides by $\dot{\theta}$ gives

$$mR_0\dot{\theta}\ddot{\theta} - mg\dot{\theta}\sin\theta = 0 \tag{47}$$

$$\frac{d}{dt}\left(\frac{1}{2}mR_0\dot{\theta}^2\right) + \frac{d}{dt}(mg\cos\theta) = 0 \tag{48}$$

$$\frac{d}{dt}\left(\frac{1}{2}mR_0\dot{\theta}^2\right) = -\frac{d}{dt}(mg\cos\theta) \tag{49}$$

$$\frac{1}{2}mR_0\dot{\theta}^2 = -mg\cos\theta + C \tag{50}$$

where C is a constant. Using the initial conditions that $\dot{\theta}(0) = \omega_0$ and $\theta(0) = \pi$, we have

$$\frac{1}{2}mR_0\omega_0^2 = -mg\cos\pi + C \tag{51}$$

$$C = \frac{1}{2}mR_0\omega_0^2 - mg \tag{52}$$

Hence we have

$$\frac{1}{2}mR_0\dot{\theta}^2 = -mg\cos\theta + \frac{1}{2}mR_0\omega_0^2 - mg \tag{53}$$

$$\dot{\theta}^2 = \omega_0^2 - \frac{2g}{R_0}(1+\cos\theta), \textbf{ or} \tag{54}$$

$$\dot{\theta} = \sqrt{\omega_0^2 - \frac{2g}{R_0}(1+\cos\theta)} \tag{55}$$

- **(1 pt)** : Knowing that a suitable integrating factor is $\dot{\theta}$, and applying it to the equation.
- **(1 pt)** : Writing down the solution for $\dot{\theta}$ up to a constant C.
- **(2 pt)** : Identifying the initial condition $\theta(0) = \pi$ and $\dot{\theta}(0) = \omega_0$. Note that if students define the z-axis to be pointing downward, then $\theta(0) = 0$ should also be accepted.
- **(1 pt)** : Correct final expression for $\dot{\theta}$ (or $\dot{\theta}^2$).

Total sub-points : 5

(f) **[2 points]** Hence find the magnitude of the force of constraint in terms of θ and ω_0.

Solution:

Substituting the expression for $\dot{\theta}^2$ from (f) to equation (42), we have

$$-mR_0\left(\omega_0^2 - \frac{2g}{R_0}(1+\cos\theta)\right) + mg\cos\theta - \lambda = 0 \tag{56}$$

$$-mR_0\omega_0^2 + 2mg(1+\cos\theta) + mg\cos\theta - \lambda = 0 \tag{57}$$

$$\lambda = 3mg\cos\theta + 2mg - mR_0\omega_0^2 \tag{58}$$

Hence, the magnitude of the force of constraint is

$$|f| = |3mg\cos\theta + 2mg - mR_0\omega_0^2| \tag{59}$$

- **(1 pt)** : Correct expression for λ.
- **(1 pt)** : Identifying λ as the magnitude of the force of constraint.

Total sub-points : 2

(g) **[3 points]** Finally, if we want the ball to be able to do a *complete round trip* around the circular track, what is the minimum value of ω_0?

Solution:

The minimum force of constraint at the top of the circular track (i.e. $\theta = 0$) is $f = 0$. So we have

$$3mg\cos 0 + 2mg - mR_0\omega_0^2 = 0 \tag{60}$$

$$mR_0\omega_0^2 = 5mg \tag{61}$$

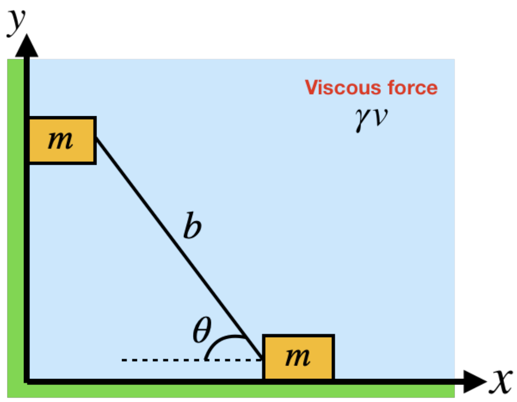
$$\omega_0 = \sqrt{\frac{5g}{R_0}} \tag{62}$$

Hence, the minimum starting angular frequency is $\omega_0 = \sqrt{\frac{5g}{R_0}}$.

- **(1 pt)** : Stating that at minimum ω_0, the force of constraint vanishes at the top of the track.
- **(1 pt)** : Using the equation for the force of constraint to solve for ω_0.
- **(1 pt)** : Correct expression for the minimum value of ω_0.

Total sub-points : 3

Problem 4 - Virtual work, generalized force and equation of motion



As shown in the figure, a system consists of two identical blocks of mass m connected by a massless rigid rod of length b with rotatable joints at the two blocks. The whole system is placed in a viscous liquid (maybe it's water). Assume that the viscous force only acts along the direction of motion of the individual masses, and has a magnitude of $f = \gamma v$, where v is the velocity of the individual masses and γ is a constant. The angle of elevation of the left mass from the right mass is θ. Take the acceleration due to gravity, which acts in the negative y direction, as g. We want to investigate the motion of the system. Assume motions only take place in the the x-y plane (i.e. There is no motion along the z direction). Neglect friction with respect to the motion of the rod.

(a) **[2 points]** How many constraint(s) is(are) there in this problem? Is the constraint holonomic or non-hologonic?

Solution:

There is only *one* constraint in the problem, which is that the distance between the two masses is a constant b, i.e.

$$f = b - \sqrt{x^2 + y^2} = \text{constant} \tag{63}$$

Since the constraint is an equation depending only on the coordinate of the system, it is a *holonomic* constraint.

- **(1 pt)** : Stating there is only one constraint in this problem.
- **(1 pt)** : Identifying that the constraint is holonomic.

Total sub-points : 2

(b) **[1 point]** Assume that the left mass can only move vertically and the right mass can only move horizontally. How many degree(s) of freedom do(es) this system have?

Solution:

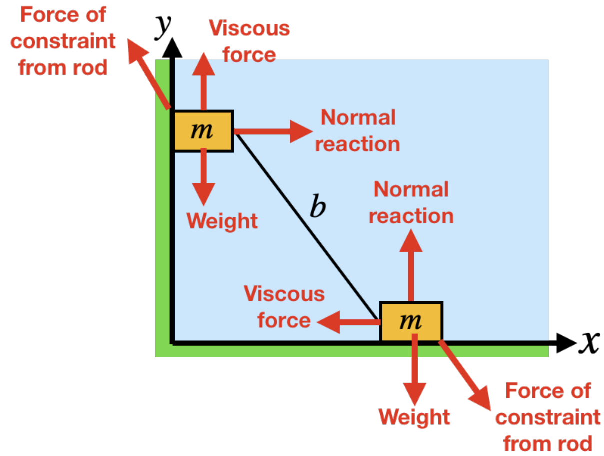
The degree of freedom for the system originally is *TWO* (since they are free to move along the x and y direction). With *ONE* holonomic constraint, the degree of freedom is reduced by *ONE*. Hence, the system only has **ONE** degree of freedom.

- **(1 pt)** : Stating that the system has only *ONE* degree of freedom.

Total sub-points : 1

(c) **[2 points]** Draw a free body diagram for the blocks in the system. Remember you also have to consider the internal forces.

Solution:



- **(0.5 pt)** : Correct pair of normal reaction.
- **(0.5 pt)** : Correct pair of weight.
- **(0.5 pt)** : Correct pair of viscous force.
- **(0.5 pt)** : Correct pair of force of constraint.

Total sub-points : 2

(d) **[1 point]** Upon a virtual displacement, which forces in (c) can do a virtual work on the system of blocks?

Solution:

Recall that forces of constraint do no virtual work. For a small displacement, only the *weight* and *viscous force* can do a virtual work on the system.

- **(0.5 pt)** : Identifying weight as a possible source of virtual work.
- **(0.5 pt)** : Identifying viscous force as a possible source of virtual work.

Total sub-points : 1

(e) **[4 points]** Let's denote the position of the left and right mass as $(0, y)$ and $(x, 0)$ respectively. Find the virtual work upon a small displacement for the system in terms of δy (for the left mass) and δx (for the right mass). Recall that x and y are *NOT* independent coordinates. In the view of this, express the virtual work in terms of $\delta\theta$ and θ only.

Solution:

In terms of δx and δy, we have

$$\delta W = \underbrace{-mg\delta y}_{\delta y<0} - \underbrace{\gamma\dot{y}\delta y}_{\dot{y},\delta y<0} - \underbrace{\gamma\dot{x}\delta x}_{\dot{x},\delta x>0} \tag{64}$$

Since x and y are not independent, in particular

$$\begin{cases} x = b\cos\theta \\ y = b\sin\theta \end{cases} \quad \text{and} \quad \begin{cases} \delta x = -b\delta\theta\sin\theta \\ \delta y = b\delta\theta\cos\theta \end{cases} \quad \text{and} \quad \begin{cases} \dot{y} = b\dot{\theta}\cos\theta \\ \dot{x} = -b\dot{\theta}\sin\theta \end{cases} \tag{65}$$

With such, we have

$$\delta W = -mg(-\cot\theta\delta x) - \gamma(b\dot{\theta}\cos\theta)(-\cot\theta\delta x) - \gamma(-b\dot{\theta}\sin\theta)\delta x \tag{66}$$

$$= (mg\cot\theta + \gamma b\dot{\theta}\cos\theta\cot\theta + \gamma b\dot{\theta}\sin\theta)\underbrace{(-b\delta\theta\sin\theta)}_{\delta x} \tag{67}$$

$$= (-mgb\cos\theta - \gamma b^2\dot{\theta})\delta\theta \tag{68}$$

- **(1 pt)** : Correct expression for virtual work in terms of δx and δy.
- **(1 pt)** : Finding δx in terms of $\delta\theta$ and θ.
- **(1 pt)** : Finding δy in terms of $\delta\theta$ and θ.
- **(1 pt)** : Finding $\dot{x}$ and $\dot{y}$ in terms of $\dot{\theta}$ and θ.
- **(1 pt)** : Correct final expression for virtual work in terms of θ and $\delta\theta$. Note: This time we dropped this item since there is a typo in the solution.

Total sub-points : 4

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(f) **[3 points]** What is(are) a (set of) suitable generalized coordinate(s) for this system? Does this agree with your answer in (b)? Also identify the generalized force for this problem in terms of the generalized coordinates you have chosen.

Solution:

Intuitively, we can see that θ constitutes a suitable generalized coordinate for this system. There is only one generalized coordinate, corresponding to *ONE* degree of freedom in the system (and hence agree with the answer in (b)).

Since the virtual work is expressed in the form

$$\delta W = Q\delta\theta \tag{69}$$

where Q is the generalized force, we have the generalized force as

$$Q = -mgb\cos\theta - \gamma b\dot{\theta} \tag{70}$$

- **(1 pt)** : Identifying θ as the suitable generalized coordinate.
- **(1 pt)** : Stating that the result agrees with the answer in (b).
- **(1 pt)** : Correct expression for the generalized force.

Total sub-points : 3

(g) **[2 points]** Using D'Alembert's principle or the Golden Rule #1 and show that the equation of motion is in the form

$$f_1(m,b,g,\gamma,\theta)\ddot{\theta} + f_2(m,b,g,\gamma,\theta)\dot{\theta} + f_3(m,b,g,\gamma,\theta) = 0 \tag{71}$$

where f_1, f_2 and f_3 are undetermined functions of m, b, g, γ and θ.

Solution:

By D'Alembert's principle or Golden Rule #1, we have

$$-mgb\cos\theta - \gamma b\dot{\theta} = mb^2\ddot{\theta} \tag{72}$$

$$\therefore mb^2\ddot{\theta} + \gamma b\dot{\theta} + mgb\cos\theta = 0 \tag{73}$$

- **(1 pt)** : Applying either D'Alembert's principle or Golden Rule #1.
- **(1 pt)** : Correct final expression (comparable to the one in the question) for the equation of motion.

Total sub-points : 2

(h) **[1 point]** If $\gamma = 0$, what is the equation of motion?

Solution:

When $\gamma = 0$, we have the equation of motion as

$$mb^2\ddot{\theta} + mgb\cos\theta = 0 \tag{74}$$

- **(1 pt)** : Correct expression for the equation of motion.

Total sub-points : 1

(i) **[10 points]** Formulate the equation of motion in part (h) (i.e. $\gamma = 0$) starting from the generalized coordinates (x, y). Apply the Lagrange's equations with Lagrange multipliers. Express the equations in terms of θ and solve for $\dot{\theta}^2$ in terms of θ. Finally, find the force of constraints in terms of θ and θ_0. Assume that $\theta = \theta_0$ initially, and the system starts from rest.

Solution:

The kinetic energy term is

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \tag{75}$$

The potential energy term is

$$V = mgy \tag{76}$$

The Lagrangian for this problem is

$$\mathcal{L} = T - V = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy \tag{77}$$

subjected to the constraint

$$f = b - \sqrt{x^2 + y^2} = 0 \tag{78}$$

Using Lagrange's equation with Lagrange multipliers, we have the equation of motion for x as

$$\frac{d}{dt}(m\dot{x}) + \lambda\frac{\partial f}{\partial x} = 0 \tag{79}$$

$$m\ddot{x} - \lambda\frac{x}{\sqrt{x^2 + y^2}} = 0 \tag{80}$$

and the equation of motion for y as

$$\frac{d}{dt}(m\dot{y}) - (-mg) + \lambda\frac{\partial f}{\partial y} = 0 \tag{81}$$

$$m\ddot{y} + mg - \lambda\frac{y}{\sqrt{x^2 + y^2}} = 0 \tag{82}$$

Using the constraint $b = \sqrt{x^2 + y^2}$, we have

$$\begin{cases} m\ddot{x} - \frac{\lambda x}{b} = 0 \\ m\ddot{y} + mg - \frac{\lambda y}{b} = 0 \end{cases} \tag{83}$$

Putting back the parametrization

$$\begin{cases} x = b\cos\theta \\ y = b\sin\theta \end{cases} \quad \text{and} \quad \begin{cases} \ddot{x} = -b\ddot{\theta}\sin\theta - b\dot{\theta}^2\cos\theta \\ \ddot{y} = b\ddot{\theta}\cos\theta - b\dot{\theta}^2\sin\theta \end{cases} \tag{84}$$

we have the equations of motion as

$$\begin{cases} -mb\ddot{\theta}\sin\theta - mb\dot{\theta}^2\cos\theta - \lambda\cos\theta = 0 \\ mb\ddot{\theta}\cos\theta - mb\dot{\theta}^2\sin\theta + mg - \lambda\sin\theta = 0 \end{cases} \tag{85}$$

and hence

$$\lambda\cos\theta = -mb\ddot{\theta}\sin\theta - mb\dot{\theta}^2\cos\theta \tag{86}$$

$$\lambda\sin\theta = mb\ddot{\theta}\cos\theta - mb\dot{\theta}^2\sin\theta + mg \tag{87}$$

Consider equation(86)$\times\sin\theta-$equation(87)$\times\cos\theta$, we have

$$0 = -mb\ddot{\theta}(\sin^2\theta + \cos^2\theta) - mg\cos\theta = 0 \tag{88}$$

$$mb\ddot{\theta} + mg\cos\theta = 0 \tag{89}$$

which is a similar equation as equation(47). Using the same technique, we have

$$mb\dot{\theta}\ddot{\theta} + mg\dot{\theta}\cos\theta = 0 \tag{90}$$

$$\frac{d}{dt}\left(\frac{1}{2}mb\dot{\theta}^2\right) = -\frac{d}{dt}(mg\sin\theta) \tag{91}$$

$$\frac{1}{2}mb\dot{\theta}^2 = -mg\sin\theta + C \tag{92}$$

$$\dot{\theta}^2 = -\frac{2g}{b}\sin\theta + C \tag{93}$$

where C is a constant. Using the initial condition that $\dot{\theta}(0) = 0$ and $\theta(0) = \theta_0$, we have

$$0 = -\frac{2g}{b}\sin\theta_0 + C \tag{94}$$

$$C = \frac{2g}{b}\sin\theta_0 \tag{95}$$

So we have

$$\dot{\theta}^2 = -\frac{2g}{b}\sin\theta + \frac{2g}{b}\sin\theta_0 \tag{96}$$

Finally, consider equation(86)$\times\cos\theta$+equation(87)$\times\sin\theta$, we have

$$\lambda(\sin^2\theta + \cos^2\theta) = -mb\dot{\theta}^2(\cos^2\theta + \sin^2\theta) + mg\sin\theta \tag{97}$$

$$\lambda = -mb\dot{\theta}^2 + mg\sin\theta \tag{98}$$

$$\lambda = -mb\left(-\frac{2g}{b}\sin\theta + \frac{2g}{b}\sin\theta_0\right) + mg\sin\theta \tag{99}$$

$$\lambda = 3mg\sin\theta - 2mg\sin\theta_0 \tag{100}$$

Hence, the force of constraint is

$$f = |3mg\sin\theta - 2mg\sin\theta_0|. \tag{101}$$

- **(1 pt)** : Correct Lagrangian in terms of x and y.
- **(1 pt)** : Correct equation of motion for x with Lagrange multiplier.
- **(1 pt)** : Correct equation of motion for y with Lagrange multiplier.
- **(2 pt)** : Converting the equations of motion into expressions involving θ and λ only; *ONE* point for $x^2 + y^2 = b^2$, and *ONE* point for transforming x, $\ddot{x}$, y and $\ddot{y}$ into functions of θ
- **(1 pt)** : Constructing a differential equation for θ only.
- **(1 pt)** : Solving $\dot{\theta}^2$ up to a constant.
- **(1 pt)** : Correct final expression for $\dot{\theta}^2$.
- **(2 pt)** : Correct expression for λ in terms of θ. Award *ONE* point if students have at least try to construct an equation for λ in terms of $\dot{\theta}^2$ and θ.

Total sub-points : 10