Physics 106a, Classical Mechanics — Fall Term, 2019
Final Exam

Do not read beyond this page until you are ready to start the exam!

Due date: Due by **7pm in Physics 106 In Box in East Bridge on Friday 13 December**. Late exams will require extenuating circumstances; otherwise, no credit will be given. The "one free extension" *cannot* be used on the final.

Material: The exam covers the material covered in Lectures 1-19.

Logistics: The exam consists of this page plus two pages of questions. Do not look at the questions until you are ready to start the exam. Please use a blue book for the exam. Alternatively, you may type your solutions on individual sheets, but the typing time must be included in the allocated time. Please review the "Honor Code and Collaboration policy" on the course website.

The exam consists of four questions, with points as indicated.
Note that many of the problems can be done by a number of different methods: if possible check your results using a different approach. And check your answers make sense physically!

Time: you may spend up to 5 hours on the exam. You may take one or two breaks, as long as you not work on the exam or consult any sources that may relate to the exam during the breaks.

References: You may consult the class text books: Hand and Finch; and Taylor. You may also consult any material on the Ph106 website for this year, including lecture notes and solutions, and notes on this class you or another student in the class have made. You may not consult any other material, including other books, any material from previous years' classes or any other classes, or any other material on the internet. You may use a calculator but not any symbolic manipulation programs (you shouldn't need them). You may use a word processor to type your solutions if you wish.

If these instruction are not clear, please consult with me before starting the exam.

Grading: The exam will count for $30\% - 35\%$ of your grade for the class.

1. **Repulsive $1/r^3$ scattering**: A fixed (infinitely massive) force center scatters a particle of mass m and initial velocity v_0 according to the force law $F(r) = k/r^3$, with $k > 0$.

   (a) (1 points) Relate the impact parameter b to the magnitude of the total angular momentum l.

   Solution:

   For an infinitely massive force center, the reduced mass $\mu = m$. The magnitude of the (constant) angular momentum is $l = mv_0 b$. Hence we have

   $$b = \frac{l}{mv_0}. \tag{1}$$

   - **(1 pt)** : Correct expression for b.

   Total sub-points : 1

   (b) (2 points) Write down an expression for the total energy, in terms of $u(\theta) = 1/r(\theta)$, and l. (For scattering problems, it makes more sense to call the polar scattering angle θ rather than thinking of it as an azimuthal angle ϕ; do you understand why?).

   Solution:

   The total energy for an orbit with potential V is

   $$E = \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} + V \tag{2}$$

   For a repulsive scattering potential, it's better to think in terms of an incoming particle at $\theta_{in} = \pi$, scattering through to an outgoing particle at infinity through an angle θ_∞. The potential for this problem is

   $$V = \frac{k}{2r^2} = \frac{k}{2}u^2 \tag{3}$$

   Thus the total energy is given by

   $$E = \frac{l^2}{2m}\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] + \frac{1}{2}ku^2. \tag{4}$$

   - **(1 pt)** : Writing down the total energy in terms of r and other parameters.
   - **(1 pt)** : Expressing the total energy in terms of u and l. If the students jump to the solution directly, full credit should still be given.

   Total sub-points : 2

   (c) (2 points) Solve for $u(\theta)$.

   Solution:

   Remember that the total energy E is a constant of motion, so $dE/d\theta = 0$. Differentiating,

   $$\frac{d^2u}{d\theta^2} + \lambda^2 u = 0 \quad \text{where} \quad \lambda^2 = 1 + \frac{km}{l^2}. \tag{5}$$

   Since $k > 0$, the solution is then

   $$u = A\sin\lambda(\pi - \theta), \tag{6}$$

   where the phase ensures that $u \to 0$ as $\theta \to \pi$ for the incoming particle.

- **(1 pt)** : Differentiating the equation for E with respect to θ.
- **(1 pt)** : Solving u.

Total sub-points : 2

(d) (2 points) Long after the collision, $r \to \infty$ or $u \to 0$, and $\theta \to \theta_\infty$. From this, derive $b(\theta_\infty)$.

Solution:

When $u \to 0$ after the collision, $\pi = \lambda(\pi - \theta_\infty)$. Thus,

$$\frac{km}{l^2} = \frac{\theta_\infty(2\pi - \theta_\infty)}{(\pi - \theta_\infty)^2}. \tag{7}$$

Use $l = mv_0 b$ to get

$$b^2 = \frac{k}{mv_0^2}\frac{(\pi - \theta_\infty)^2}{(2\pi - \theta_\infty)\theta_\infty}. \tag{8}$$

- **(1 pt)** : Using $\pi = \lambda(\pi - \theta_\infty)$ to relate θ_∞ with k, m, and l.
- **(1 pt)** : Using Expressing b in terms of θ_∞.

Total sub-points : 2

(e) (3 points) Show that the differential scattering cross section (with $\theta = \theta_\infty$) is

$$\frac{d\sigma}{d\Omega} = \frac{k\pi^2(\pi - \theta)}{mv_0^2\theta^2(2\pi - \theta)^2 \sin\theta}.$$

(You might want to plot it, but not for any extra credit).

Solution:

The differential scattering cross-section is defined and derived in Lecture 9:

$$\frac{d\sigma}{d\Omega} = \left|\frac{2\pi b db}{2\pi \sin\theta d\theta}\right| = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| \tag{9}$$

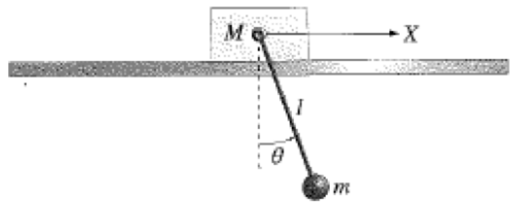
Differentiation of $b(\theta)$ gives

$$\frac{d\sigma}{d\Omega} = \frac{1}{2\sin\theta}\left|2b\frac{db}{d\theta}\right| = \frac{k}{mv_0^2}\frac{\pi^2(\pi - \theta)}{\theta^2(2\pi - \theta)^2 \sin\theta}. \tag{10}$$

- **(1 pt)** : Writing down the expression for the differential cross section in terms of b and θ.
- **(1 pt)** : Substituting $b(\theta)$ into the differential cross section equation.
- **(1 pt)** : Correct final expression for the differential cross section.

Total sub-points : 3

2. **Small Oscillations and normal modes**
A simple pendulum of mass m and length l is attached to a block of mass M that is free to move without friction along a horizontal track. As shown in the figure, there are two degrees of freedom, x and θ. (If your m's and M's look too much alike, call them m_p and m_b, respectively.)



(a) (2 points) Write down the Lagrangian in both full generality, and then in the limit of small amplitude motion.

Solution:

The Lagrangian is given by

$$\begin{aligned} L &= T - V \quad (11)\\ &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left[v_x^2 + v_y^2\right] - mgl(1-\cos\theta) \quad (12)\\ &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left[(\dot{x} + l\dot{\theta}\cos\theta)^2 + (l\dot{\theta}\sin\theta)^2\right] - mgl(1-\cos\theta) \quad (13)\end{aligned}$$

In the small amplitude limit, $\theta \ll 1$, we will keep terms only up to the order θ^2, $\theta\dot{\theta}$, $\dot{\theta}^2$:

$$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + 2l\dot{x}\dot{\theta} + l^2\dot{\theta}^2\right) - \frac{1}{2}mgl\theta^2. \quad (14)$$

- **(1 pt)** : Correct general Lagrangian.
- **(1 pt)** : Correct Lagrangian for small amplitude motion.

Total sub-points : 2

(b) (1 points) Find the linearized equations of motion.

Solution:

By using the Lagrange equation, we have the equation of motion for x as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \quad (15)$$

$$\frac{d}{dt}((M+m)\dot{x} + ml\dot{\theta}) = 0 \quad (16)$$

$$(M+m)\ddot{x} + ml\ddot{\theta} = 0 \quad (17)$$

and the equation of motion for θ as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \quad (18)$$

$$ml^2\ddot{\theta} + ml\ddot{x} = -mgl\theta \quad (19)$$

(Note: One can integrate the first EOM to get $(M+m)x + ml\theta = At + B$, where A and B are constants of integration associated with initial conditions.)

- **(0.5 pt)** : Correct equation of motion for x.
- **(0.5 pt)** : Correct equation of motion for θ.

Total sub-points : 1

(c) (3 points) Find the normal mode frequencies.

Solution:

The T and V energy matrices are

$$\mathbf{T} = \begin{bmatrix} M+m & ml \\ ml & ml^2 \end{bmatrix} \tag{20}$$

and

$$\mathbf{V} = \begin{bmatrix} 0 & 0 \\ 0 & mgl \end{bmatrix} \tag{21}$$

To solve for the normal mode frequencies, we consider the characteristic equation

$$|\mathbf{V} - \omega^2\mathbf{T}| = 0 \tag{22}$$

We have

$$\begin{vmatrix} -\omega^2(M+m) & -\omega^2 ml \\ -\omega^2 ml & mgl - \omega^2 ml^2 \end{vmatrix} = 0 \tag{23}$$

$$-\omega^2 \begin{vmatrix} M+m & -\omega^2 ml \\ ml & mgl - \omega^2 ml^2 \end{vmatrix} = 0 \tag{24}$$

$$-\omega^2(mMgl + m^2gl - \omega^2 mMl^2 - \omega^2 m^2l^2 + \omega^2 m^2 l^2) = 0 \tag{25}$$

$$\omega^2[Ml\omega^2 - g(M+m)] = 0 \tag{26}$$

Hence we have the normal mode frequencies as

$$\omega^2 = 0 \quad \text{or} \quad \frac{M+m}{M}\frac{g}{l} \tag{27}$$

(Note: One can check that if $M \gg m$, we have an infinitely heavy block which does not move, and the only thing that moves is the pendulum with $\omega = \sqrt{g/l}$. If $M \ll m$, the block is free to slide back and forth to keep the CM motionless, so the pendulum's θ motion is mostly due to the block motion, not the bob motion.)

- **(1 pt)** : Correct **T** and **V** energy matrices. Note that if students use the **Hand and Finch** definition, credit should still be given.
- **(2 pt)** : Correct normal mode frequencies. Partial credit can be given for correct steps.

Total sub-points : 3

(d) (3 points) Find the normal mode vectors of the system.

Solution:

There are two normal mode frequencies: $\omega_A^2 = 0$ and $\omega_B^2 = \frac{M+m}{M}\frac{g}{l}$.

- For $\omega_A^2 = 0$, we have from the characteristic equation, that

$$\begin{bmatrix} 0 & 0 \\ 0 & mgl \end{bmatrix} \begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{28}$$

$$\theta = 0 \tag{29}$$

Normalizing the normal mode vector, we have $x = 1$. Hence the normal mode vector associated with ω_A is

$$\vec{v_A} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{30}$$

- For $\omega_B^2 = \frac{M+m}{M}\frac{g}{l}$, we have from the characteristic equation, that

$$\begin{bmatrix} -\frac{(M+m)^2}{M}\frac{g}{l} & -\frac{M+m}{M}mg \\ -\frac{M+m}{M}mg & -\frac{m^2gl}{M} \end{bmatrix} \begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{31}$$

$$x = -\frac{ml}{M+m}\theta \tag{32}$$

This give a normal mode vector of

$$\vec{v_B} = \begin{bmatrix} -\frac{ml}{m+M} \\ 1 \end{bmatrix}. \tag{33}$$

- **(1.5 pt)** : Correct normal mode vector for $\omega_A^2 = 0$.
- **(1.5 pt)** : Correct normal mode vector for $\omega_B^2 = \frac{M+m}{M}\frac{g}{l}$.

Total sub-points : 3

(e) (1 points) Discuss the physical interpretation of the normal mode frequencies and eigenvectors.

Solution:

The normal mode vector for $\omega_A^2 = 0$ is

$$\vec{v_A} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{34}$$

This means that there is no θ-motion. The system is purely moving uniformly along the x-direction (purely rectilinear).

The normal mode vector for $\omega_B^2 = \frac{M+m}{M}\frac{g}{l}$ is

$$\vec{v_B} = \begin{bmatrix} -\frac{ml}{m+M} \\ 1 \end{bmatrix}. \tag{35}$$

Both the motions for the x and θ coordinates are oscillatory (out of phase). This means that the upper block and the pendulum are simply oscillating about a common point. The block will have a finite and bounded displacement along the x-direction. To further see this, consider the center-of-mass position of the system:

$$(M+m)x_{CM} = m(l\theta + x) + Mx = ml\theta + (M+m)x \tag{36}$$

$$= -ml\frac{M+m}{ml}A_B\cos(\omega_B t + \phi_B) + (M+m)A_B\cos(\omega_B t + \phi_B) \tag{37}$$

$$= 0 \tag{38}$$

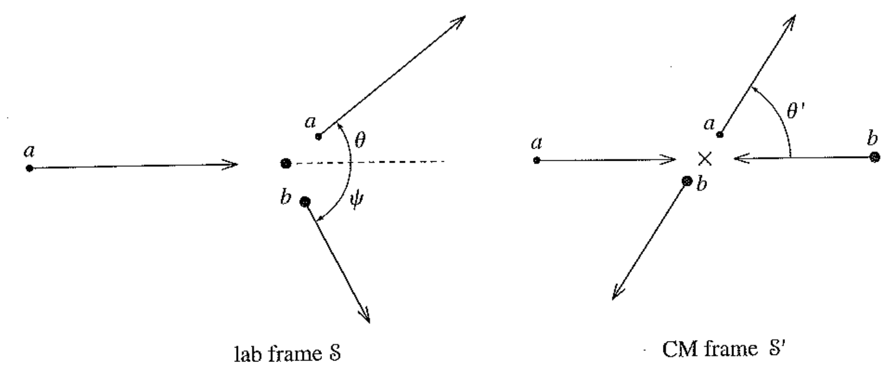
which says that the center-of-mass is not moving in this mode, meaning that the motion of this mode is purely oscillatory.

- **(0.5 pt)** : Stating that the motion for $\omega_A^2 = 0$ is purely rectilinear (or similar wordings).
- **(0.5 pt)** : Stating that the motion for $\omega_B^2 = \frac{M+m}{M}\frac{g}{l}$ is purely oscillatory (or similar wordings).

Total sub-points : 1

Post-question discussion: There are two degrees of freedom, x and θ. So expect two normal modes of oscillation. We can guess that one will correspond to $\dot{x}$ = const, $\theta = \dot{\theta} = 0$, uniform translation, so $\omega = 0$. The other will be an out-of-phase oscillation in which $\dot{x} > 0$ when $\dot{\theta} < 0$ and vice versa.

3. **Geometry of elastic collisions:** Consider the relativistic collision shown in the figure:



(CM is the center of momentum frame). In the lab frame the incoming particle has energy E_a and momentum $\vec{p}_a$.

(a) (3 points). Show that the velocity of the CM frame S' relative to the lab frame S is

$$\vec{V} = \frac{\vec{p}_a}{E_a + m_b}$$

Solution:
The four-momenta for particle a and b in the CM frame are given by

$$\begin{cases} P_a'^{\mu} = (E_a', \vec{p}_a{}') \\ P_b'^{\mu} = (E_b', \vec{p}_b{}') \end{cases} \tag{39}$$

and those in the lab frame is given by

$$\begin{cases} P_a^{\mu} = (E_a, \vec{p}_a) \\ P_b^{\mu} = (m_b, 0) \quad (b \text{ is not moving in the lab frame.}) \end{cases} \tag{40}$$

In the CM frame, the total linear momentum is 0, which gives

$$\vec{p}_a{}' = -\vec{p}_b{}' \tag{41}$$

The total four-momentum in the CM frame is hence

$$P_{\text{total}}'^{\mu} = P_a'^{\mu} + P_b'^{\mu} = (E_a' + E_b', 0) \tag{42}$$

and the total four-momentum in the lab frame is given by

$$P^{\mu}_{\text{total}} = P^{\mu}_{a} + P^{\mu}_{b} = (E_a + m_b, \vec{p}_a) \tag{43}$$

Relating P^{μ}_{total} and P'^{μ}_{total} with Lorentz transformation, we have

$$\begin{bmatrix} E'_a + E'_b \\ 0 \end{bmatrix} = \begin{bmatrix} \gamma & -V\gamma \\ -V\gamma & \gamma \end{bmatrix} \begin{bmatrix} E_a + m_b \\ \vec{p}_a \end{bmatrix} \tag{44}$$

Consider the spatial element, we have

$$0 = -V\gamma(E_a + m_b) + \gamma\vec{p}_a \tag{45}$$

$$V = \frac{\vec{p}_a}{E_a + m_b} \tag{46}$$

- **(1 pt)** : Formulating the total four-momenta in the CM frame and lab frame.
- **(1 pt)** : Connecting the total four-momenta in the CM frame and lab frame with Lorentz transformation.
- **(1 pt)** : Obtaining the correct final expression by considering the spatial element of the total four-momenta.

Total sub-points : 3

(b) (3 points). By transforming between the two frames, show that

$$\tan\theta = \frac{\sin\theta'}{\gamma_V(\cos\theta' + V/v'_a)}$$

with v'_a the speed of a in the CM frame, and $\gamma_V = (1 - V^2)^{-1/2}$.

Solution:

The four-momenta for particle a in the CM frame is

$$P'^{\mu}_{a} = (E'_a, {p_a}' \cos\theta', {p_a}' \sin\theta') \tag{47}$$

and that in the lab frame is given by

$$P^{\mu}_{a} = (E_a, p_a \cos\theta, p_a \sin\theta) \tag{48}$$

Relating P^{μ}_{a} and P'^{μ}_{a} with Lorentz transformation, we have

$$\begin{bmatrix} E_a \\ p_a \cos\theta \\ p_a \sin\theta \end{bmatrix} = \begin{bmatrix} \gamma & V\gamma & 0 \\ V\gamma & \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} E'_a \\ {p_a}' \cos\theta' \\ {p_a}' \sin\theta' \end{bmatrix} \tag{49}$$

Consider the x and y elements, we have

$$\begin{cases} p_a \cos\theta = V\gamma E'_a + \gamma {p_a}' \cos\theta' \\ p_a \sin\theta = {p_a}' \sin\theta' \end{cases} \tag{50}$$

With the above two equations, we have

$$\tan\theta = \frac{{p_a}'\sin\theta'}{\gamma({p_a}'\cos\theta' + VE'_a)} \tag{51}$$

Now, recall that

$$\begin{aligned}
E &= \gamma_u m && (52)\\
E &= \gamma_u\sqrt{E^2 - p^2} && (53)\\
E^2 &= \gamma_u^2(E^2 - p^2) && (54)\\
E^2(\gamma_u^2 - 1) &= \gamma_u^2 p^2 && (55)\\
E^2 &= \frac{\gamma_u^2}{\gamma_u^2 - 1}p^2 && (56)\\
E^2 &= \frac{1}{u^2}p^2 && (57)\\
E &= \frac{1}{u}p && (58)
\end{aligned}$$

where γ_u is the Lorentz factor calculated using the speed u of the particle. With such, we have

$$\begin{aligned}
\tan\theta &= \frac{{p_a}'\sin\theta'}{\gamma({p_a}'\cos\theta' + VE'_a)} && (59)\\
&= \frac{{p_a}'\sin\theta'}{\gamma({p_a}'\cos\theta' + \frac{V{p_a}'}{v'_a})} && (60)\\
&= \frac{\sin\theta'}{\gamma(\cos\theta' + \frac{V}{v'_a})} && (61)
\end{aligned}$$

- **(0.5 pt)** : Formulating the four-momenta of particle a in the CM frame and the lab frame.
- **(0.5 pt)** : Relating the two 4-momenta using Lorentz transformation.
- **(1 pt)** : Obtaining the intermediate expression by considering the x and y spatial elements.
- **(1 pt)** : Obtaining the final expression by considering the relationship between E'_a and v'_a.

Total sub-points : 3

Now specialize to the case of $m_a = m_b$.

(c) (3 points). Show that in this case $V/v'_a = 1$, and find an expression for $\tan\psi$ analogous to the one above for $\tan\theta$.

Solution:

Consider the four-momenta of particle b. In the lab frame we have

$$P_b^\mu = (m_b, 0) \tag{62}$$

and in the CM frame we have

$$P_b'^{\mu} = (E_b', p_b') \tag{63}$$

With Lorentz transformation, we have

$$\begin{bmatrix} E_b' \\ p_b' \end{bmatrix} = \begin{bmatrix} \gamma & -V\gamma \\ -V\gamma & \gamma \end{bmatrix} \begin{bmatrix} m_b \\ 0 \end{bmatrix} \tag{64}$$

and hence

$$\begin{cases} E_b' = \gamma m_b \\ p_b' = -V\gamma m_b \end{cases} \tag{65}$$

As we have shown in the previous part, we have

$$\frac{E_b'}{p_b'} = \frac{1}{v_b'} = -\frac{1}{V} \tag{66}$$

$$v_b' = -V \tag{67}$$

Since in the CM frame, $v_a' = -v_b'$, we have

$$v_a' = V \tag{68}$$

$$\frac{V}{v_a'} = 1 \tag{69}$$

Now, the four-momenta for particle b in the CM frame is

$$P_b'^{\mu} = (E_b', -{p_b}' \cos\theta', -{p_a}' \sin\theta') \tag{70}$$

and that in the lab frame is given by

$$P_b^{\mu} = (E_b, p_b \cos\psi, -p_b \sin\psi) \tag{71}$$

Relating P_b^{μ} and $P_b'^{\mu}$ with Lorentz transformation, we have

$$\begin{bmatrix} E_b \\ p_b \cos\psi \\ -p_b \sin\psi \end{bmatrix} = \begin{bmatrix} \gamma & V\gamma & 0 \\ V\gamma & \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} E_b' \\ -{p_b}' \cos\theta' \\ -{p_b}' \sin\theta' \end{bmatrix} \tag{72}$$

Consider the x and y elements, we have

$$\begin{cases} p_b \cos\psi = V\gamma E_b' - \gamma {p_b}' \cos\theta' \\ -p_b \sin\psi = -{p_b}' \sin\theta' \end{cases} \tag{73}$$

With the above two equations, we have

$$-\tan\psi = \frac{-p_b{}'\sin\theta'}{\gamma(-p_b'\cos\theta' + VE_b')} \tag{74}$$

$$= \frac{-p_b{}'\sin\theta'}{\gamma(-p_b'\cos\theta' + \frac{|Vp_b'|}{|v_b'|})} \tag{75}$$

$$= -\frac{\sin\theta'}{\gamma(-\cos\theta' + \frac{|V|}{|v_b'|})} \tag{76}$$

$$= -\frac{\sin\theta'}{\gamma(-\cos\theta' + 1)} \quad (|V| = |v_b'|) \tag{77}$$

$$-\tan\psi = \frac{\sin\theta'}{\gamma(\cos\theta' - 1)} \tag{78}$$

$$\tan\psi = -\frac{\sin\theta'}{\gamma(\cos\theta' - 1)} \tag{79}$$

- **(0.5 pt)** : Relating the four-momenta for b in the lab frame and CM frame using Lorentz transformation.
- **(1 pt)** : Showing that $v_b' = -V$, and hence $v_a' = V$ (i.e. $V/v_a' = 1$).
- **(1.5 pt)** : Using similar steps as in part (c) to obtain an expression for $\tan\psi$.

Total sub-points : 3

(d) (3 points). Show that the angle between the two outgoing momenta is

$$\tan(\theta + \psi) = \frac{2}{\gamma_V V^2 \sin\theta'}$$

and that in the nonrelativistic limit of all speeds small compared with the speed of light this reduces to the well known result for the elastic collision of equal mass particles $\theta + \psi = 90°$.

Solution:

We have

$$\tan\theta = \frac{\sin\theta'}{\gamma(\cos\theta' + 1)} \tag{80}$$

and

$$\tan\psi = -\frac{\sin\theta'}{\gamma(\cos\theta' - 1)} \tag{81}$$

Using the trigonometric ratio

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}, \tag{82}$$

we have

$$\tan(\theta+\psi) = \frac{\tan\theta + \tan\psi}{1-\tan\psi\tan\theta} \tag{83}$$

$$= \frac{\frac{\sin\theta'}{\gamma(\cos\theta'+1)} - \frac{\sin\theta'}{\gamma(\cos\theta'-1)}}{1-\left(\frac{\sin\theta'}{\gamma(\cos\theta'+1)}\right)\left(-\frac{\sin\theta'}{\gamma(\cos\theta'-1)}\right)} \tag{84}$$

$$= \frac{\frac{\sin\theta'}{\gamma}\left(\frac{2}{1-\cos^2\theta'}\right)}{1-\frac{\sin^2\theta'}{\gamma^2}\frac{1}{1-\cos^2\theta'}} \tag{85}$$

$$= \frac{\frac{\sin\theta'}{\gamma}\left(\frac{2}{\sin^2\theta'}\right)}{1-\frac{\sin^2\theta'}{\gamma^2}\frac{1}{\sin^2\theta'}} \tag{86}$$

$$= \frac{\frac{2}{\gamma\sin\theta'}}{1-\frac{1}{\gamma^2}} \tag{87}$$

$$= \frac{\frac{2}{\gamma\sin\theta'}}{V^2} \tag{88}$$

$$= \frac{2}{\gamma V^2\sin\theta'} \tag{89}$$

In the non-relativistic limit, $V \to 0$ and $\gamma \to 1$. We have

$$\tan(\theta+\psi) \to \infty \tag{90}$$

which gives

$$\theta+\psi = 90^{\text{o}}. \tag{91}$$

- **(1 pt)** : Using the trigonometric ratio for $\tan(A+B)$.
- **(1 pt)** : Correct final expression for $\tan(\theta+\psi)$.
- **(1 pt)** : Showing that $\theta+\psi = 90^{\text{o}}$ in the non-relativistic limit.

Total sub-points : 3

4. **Pion production from the microwave background:** Consider the reaction between a nucleon N and the background microwave radiation approximated as a sea of photons with temperature $T = 3\text{K}$ and typical energy $E_\gamma = k_B T \simeq 2.5\times10^{-10}$ MeV.

$$\gamma + N \to N + \pi$$

with the neutron mass-energy $m_N = 940$ MeV$/c^2$, and pion mass $m_\pi = 140$ MeV$/c^2$ (but use $c = 1$ here).

(a) (4 points). Using 4-vectors, find an equation giving the threshold energy E_N of the nucleon for this reaction to occur in terms of the given parameters, assuming a head on collision (photon and nucleon moving in opposite directions).
Solution:

The four-momenta (in the lab frame) for the particles involved in this problem are

$$P_\gamma = (E_\gamma, p_\gamma) \tag{92}$$
$$P_N^i = (E_N, p_N) \tag{93}$$
$$P_N^f = (E_N^f, p_N^f) \tag{94}$$
$$P_\pi = (E_\pi, p_\pi) \tag{95}$$
$$\tag{96}$$

By conservation of four-momentum, we have

$$P_\gamma + P_N^i = P_N^f + P_\pi \tag{97}$$
$$\tag{98}$$

Squaring both sides, we have

$$(P_\gamma + P_N^i)^2 = (P_N^f + P_\pi)^2 \tag{99}$$

Consider the RHS. This is the norm of the total final four-momentum. If we evaluate this dot-product (remember, dot products are invariant) in the COM frame, since at threshold the products are not moving, we simply have

$$(P_N^f + P_\pi)^2 = [(m_N, 0) + (m_\pi, 0)]^2 \tag{100}$$
$$= [(m_N + m_\pi, 0)]^2 \tag{101}$$
$$= (m_N + m_\pi)^2 \tag{102}$$

Now, the LHS can be expanded to

$$(P_\gamma + P_N^i)^2 = P_\gamma^2 + (P_N^i)^2 + 2P_\gamma \cdot P_N^i \tag{103}$$

The norm of four-momentum gives the rest mass energy squared of the particle. Hence we have $P_\gamma^2 = 0$ and $(P_N^i)^2 = m_N^2$. For a head on collision, we have, in the lab frame, that p_γ and p_N are opposite in direction. Hence, we have

$$P_\gamma \cdot P_N^i = E_\gamma E_N + p_N p_\gamma \tag{104}$$
$$= E_\gamma E_N + p_N E_\gamma \tag{105}$$
$$= E_\gamma E_N + E_\gamma \sqrt{E_N^2 - m_N^2} \tag{106}$$

Finally, we get

$$m_N^2 + 2[E_\gamma E_N + E_\gamma \sqrt{E_N^2 - m_N^2}] = (m_N + m_\pi)^2 \tag{107}$$
$$E_N + \sqrt{E_N^2 - m_N^2} = \frac{(m_N + m_\pi)^2 - m_N^2}{2E_\gamma} \tag{108}$$
$$\sqrt{E_N^2 - m_N^2} = \frac{(m_N + m_\pi)^2 - m_N^2}{2E_\gamma} - E_N \tag{109}$$
$$E_N^2 - m_N^2 = \left(\frac{(m_N + m_\pi)^2 - m_N^2}{2E_\gamma}\right)^2 + E_N^2 - 2E_N \frac{(m_N + m_\pi)^2 - m_N^2}{2E_\gamma} \tag{110}$$
$$2E_N \frac{(m_N + m_\pi)^2 - m_N^2}{2E_\gamma} = \left(\frac{(m_N + m_\pi)^2 - m_N^2}{2E_\gamma}\right)^2 + m_N^2 \tag{111}$$
$$E_N = \frac{(m_N + m_\pi)^2 - m_N^2}{4E_\gamma} + \frac{E_\gamma m_N^2}{(m_N + m_\pi)^2 - m_N^2} \tag{112}$$

- **(0.5 pt)** : Writing down the conservation of four momentum.
- **(1 pt)** : Squaring both sides, and evaluate the RHS in the COM frame to give $(m_N + m_\pi)^2$.
- **(0.5 pt)** : Evaluating the LHS in the lab frame, and obtain $(P_N^i)^2 = m_N^2$ and $P_\gamma^2 = 0$.
- **(1 pt)** : Evaluating $P_N^i \cdot P_\gamma$ in the lab frame, and expressing the answer in terms of E_γ, E_N and m_N.
- **(1 pt)** : Correct final expression for E_N. Note that some students may further simplify their expression. Credit should still be given as long as their expressions are equivalent to the suggested solution.

Total sub-points : 4

(b) (1 point). Evaluate E_N in MeV to one significant figure using the approximation $E_N \gg m_N$.

Solution:

In the limit $E_N \gg m_N$, we have

$$E_N = \frac{(m_N + m_\pi)^2 - m_N^2}{4E_\gamma} + \frac{E_\gamma m_N^2}{(m_N + m_\pi)^2 - m_N^2} \tag{113}$$

$$\approx \frac{(m_N + m_\pi)^2 - m_N^2}{4E_\gamma} \tag{114}$$

$$= \frac{(940 + 140)^2 - 940^2}{4 \times 2.5 \times 10^{-10}} \tag{115}$$

$$\approx 3 \times 10^{14} \text{ MeV}. \tag{116}$$

- **(0.5 pt)** : Correct approximation for E_N.
- **(0.5 pt)** : Correct value for E_N.

Total sub-points : 1

5. **Pion decay**: A positive pion (mass m_π) decays into a positive muon (mass m_μ) and a neutrino (essentially zero mass), $\pi^+ \to \mu^+ + \nu$.

(a) (3 points). If the pion is at rest, compute the total energy, kinetic energy, momentum, and speed, of the outgoing muon. (Simplify all the expressions!) (If you are curious, you may use a calculator to compute what fraction of the speed of light this is, using $m_\pi = 140$ MeV and $m_\mu = 106$ MeV.)

Solution:

Let's work in the center of momentum frame (here, the COM frame coincides with the lab frame since linear momentum has to be conserved, and initially the linear momentum is zero).

Initially, the four momentum is

$$P_i^\pi = (m_\pi, 0, 0, 0) \tag{117}$$

After the disintegration, we have

$$P_f = P_f^\mu + P_f^\nu \tag{118}$$

By conservation of four momentum, we have

$$\begin{aligned} P_i^\pi &= P_f^\mu + P_f^\nu \qquad (119)\\ P_f^\nu &= P_i^\pi - P_f^\mu \qquad (120)\\ |P_f^\nu|^2 &= |P_i^\pi|^2 + |P_f^\mu|^2 - 2P_i^\pi \cdot P_f^\mu \qquad (121) \end{aligned}$$

We have $|P|^2 = m^2$. Since neutrino is massless, we have $|P_f^\mu|^2 = 0$. Also, $|P_i|^2 = |P_i^\pi|^2 = m_\pi^2$ and $|P_f^\mu|^2 = m_\mu^2$. We have

$$0 = m_\pi^2 + m_\mu^2 - 2P_i \cdot P_f^\mu \tag{122}$$

In the COM frame, we have

$$P_f^\mu = (E_\mu, p_\mu) \tag{123}$$

Hence, we have

$$P_i \cdot P_f^\mu = m_\pi E_\mu \tag{124}$$

and hence

$$\begin{aligned} 0 &= m_\pi^2 + m_\mu^2 - 2m_\pi E_\mu \qquad (125)\\ E_\mu &= \frac{m_\pi^2 + m_\mu^2}{2m_\pi} \qquad (126) \end{aligned}$$

which gives the total energy of the muon. Note that the total energy E_μ is a combination of the kinetic energy T_μ and rest mass energy m_μ. Hence we have

$$E_\mu = T_\mu + m_\mu \tag{127}$$

and thus the kinetic energy of the muon is

$$\begin{aligned} T_\mu + m_\mu &= \frac{m_\pi^2 + m_\mu^2}{2m_\pi} \qquad (128)\\ T_\mu &= \frac{m_\pi^2 + m_\mu^2}{2m_\pi} - m_\mu \qquad (129)\\ T_\mu &= \frac{m_\pi^2 - 2m_\mu m_\pi + m_\mu^2}{2m_\pi} \qquad (130)\\ T_\mu &= \frac{(m_\pi - m_\mu)^2}{2m_\pi} \qquad (131) \end{aligned}$$

The momentum of the muon is given by

$$p_\mu = \sqrt{E_\mu^2 - m_\mu^2} \tag{132}$$

$$= \sqrt{\left(\frac{m_\pi^2 + m_\mu^2}{2m_\pi}\right)^2 - m_\mu^2} \tag{133}$$

$$= \sqrt{\left(\frac{m_\pi^2 + m_\mu^2}{2m_\pi} + m_\mu\right)\left(\frac{m_\pi^2 + m_\mu^2}{2m_\pi} - m_\mu\right)} \tag{134}$$

$$= \sqrt{\frac{(m_\pi + m_\mu)^2(m_\pi - m_\mu)^2}{4m_\pi^2}} \tag{135}$$

$$= \frac{(m_\pi + m_\mu)(m_\pi - m_\mu)}{2m_\pi} \tag{136}$$

$$= \frac{m_\pi^2 - m_\mu^2}{2m_\pi} \tag{137}$$

and finally the speed of the muon is given by

$$u_\mu = \frac{p_\mu}{E_\mu} \tag{138}$$

$$= \frac{m_\pi^2 - m_\mu^2}{2m_\pi} \div \frac{m_\pi^2 + m_\mu^2}{2m_\pi} \tag{139}$$

$$= \frac{m_\pi^2 - m_\mu^2}{m_\pi^2 + m_\mu^2} \tag{140}$$

- **(1 pt)** : Correct total energy E_μ of muon. Partial credit can be given for correct steps.
- **(1 pt)** : Correct kinetic energy T_μ of muon.
- **(0.5 pt)** : Correct momentum p_μ of muon.
- **(0.5 pt)** : Correct speed u_μ of muon.

Total sub-points : 3

(b) (2 points). If the pion is moving with a total energy of E_π MeV, what are the minimum and maximum energies possible for the outgoing muon?

Solution:

Let's again work in the center of momentum frame. Assume that the initial pion is moving along the x direction, we have the initial four momentum (in the pion / COM frame) as

$$P_\pi^{\text{COM}} = (E_\pi, 0, 0, 0) \tag{141}$$

After the disintegration, the muon and neutrino will move at an angle relative to the x axis. Let the angle between the x-axis and the path of muon be θ in the COM frame. We have

$$P_\mu^{\text{COM}} = (E_\mu, p_\mu \cos\theta, p_\mu \sin\theta) \tag{142}$$

where E_μ and p_μ have the same expressions as in part (a) (remember, here we are also doing the calculations in the COM frame, as in part (a)).

The Lorentz transformation from the COM frame to the lab frame gives

$$\begin{bmatrix} E_\mu^{\text{lab}} \\ p_{\mu,x}^{\text{lab}} \\ p_{\mu,y}^{\text{lab}} \end{bmatrix} = \begin{bmatrix} \gamma & V_\pi\gamma \\ V_\pi\gamma & \gamma \end{bmatrix} \begin{bmatrix} E_\mu \\ p_\mu \cos\theta \\ p_\mu \sin\theta \end{bmatrix} \tag{143}$$

For the *time* element, we have

$$E_\mu^{\text{lab}} = \gamma E_\mu + V_\pi \gamma p_\mu \cos\theta \tag{144}$$

The total energy of the pion is related to its rest mass by

$$E_\pi = \gamma m_\pi \tag{145}$$

$$\gamma = \frac{E_\pi}{m_\pi} \tag{146}$$

and its speed V_π can be found by

$$V_\pi = \frac{p_\pi}{E_\pi} = \frac{\sqrt{E_\pi^2 - m_\pi^2}}{E_\pi} \tag{147}$$

With these and the answers from part (a), we have

$$E_\mu^{\text{lab}} = \frac{E_\pi E_\mu}{m_\pi} + \frac{\sqrt{E_\pi^2 - m_\pi^2}}{E_\pi} \frac{E_\pi}{m_\pi} p_\mu \cos\theta \tag{148}$$

$$= \frac{E_\pi}{m_\pi} \frac{m_\pi^2 + m_\mu^2}{2m_\pi} + \frac{\sqrt{E_\pi^2 - m_\pi^2}}{m_\pi} \left(\frac{m_\pi^2 - m_\mu^2}{2m_\pi} \right) \cos\theta \tag{149}$$

The maximum energy is obtained when $\theta = 0$, i.e.

$$E_{\mu,\max}^{\text{lab}} = \frac{E_\pi}{m_\pi} \frac{m_\pi^2 + m_\mu^2}{2m_\pi} + \frac{\sqrt{E_\pi^2 - m_\pi^2}}{m_\pi} \left(\frac{m_\pi^2 - m_\mu^2}{2m_\pi} \right) \tag{150}$$

and the minimum energy is obtained when $\theta = \pi$, i.e.

$$E_{\mu,\min}^{\text{lab}} = \frac{E_\pi}{m_\pi} \frac{m_\pi^2 + m_\mu^2}{2m_\pi} - \frac{\sqrt{E_\pi^2 - m_\pi^2}}{m_\pi} \left(\frac{m_\pi^2 - m_\mu^2}{2m_\pi} \right) \tag{151}$$

(Post-exam discussion: If one puts in the value $E_\pi = 640$ MeV, $m_\pi = 140$ MeV/c^2 and $m_\mu = 106$ MeV/c^2, we will get $E_{\mu,\max}^{\text{lab}} = 637$ MeV, and $E_{\mu,\min}^{\text{lab}} = 370$ MeV.)

- **(1 pt)** : Relating the four-momentum of muon in the COM frame and lab frame by Lorentz transformation. Students should also have formulated correct expressions (with angles) for the four-momenta of muon in the COM frame and lab frame.
- **(0.5 pt)** : Correct final expression for $E_{\mu,\max}^{\text{lab}}$.
- **(0.5 pt)** : Correct final expression for $E_{\mu,\min}^{\text{lab}}$.

Total sub-points : 3