Physics 106a: Assignment 7 (Solutions prepared by Alvin Li)

22 November, 2019; due 7pm Friday, 6 December (**TWO WEEKS**) in the "Ph106 In Box" in East Bridge mailbox. The final exam will be posted on Friday, 6 December; due Friday, 13 December.

Hamiltonian Formulation, Special Relativity

Reading

Hamiltonian mechanics: Hand and Finch Chapter 5. Special Relativity: Hand and Finch Chapter 12.

Problems

1. **Spherical pendulum:** This is the standard pendulum of a mass m on a string of length l in gravity g but considering the full range of motion on the surface of the sphere of radius l assuming the string remains taut. Use angles θ of the string to the vertical, with $\theta = 0$ the rest state, and ϕ the angle about the horizontal. Hand and Finch treats this problem using the Hamiltonian in Example 3 of §5.5 (I use l instead of R for the length of the string). You will use the Routhian approach.

   (a) Preliminaries: write down the Lagrangian and hence the momenta p_θ, p_phi conjugate to θ, ϕ; show that the coordinate ϕ is ignorable (cyclic).

   [Solution]

   The Lagrangian is given by

$$\mathcal{L} = \underbrace{\frac{1}{2}ml^2\dot{\theta}^2 + \frac{1}{2}ml^2\sin^2\theta\dot{\phi}^2}_{\text{K.E.}} - \underbrace{mgl(1-\cos\theta)}_{\text{P.E.}} \tag{1}$$

   The conjugate momentum p_i for the generalized coordinate q_i is given by

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \tag{2}$$

   Hence, we have

$$\begin{cases} p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = ml^2\dot{\theta} \\ p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = ml^2\sin^2\theta\dot{\phi} \end{cases} \tag{3}$$

   Note that L does not depend explicitly on phi. Hence, we say that phi is cyclic (ignorable) and thus its conjugate momentum p_ϕ is a **conserved quantity**.

   - **(1 pt)** : Correct expression for the Lagrangian.
   - **(1 pt)** : Correct conjugate momentum p_θ.
   - **(1 pt)** : Correct conjugate momentum p_ϕ.
   - **(1 pt)** : Correct explanation (argument) for ϕ being an ignorable coordinate.

   Total sub-points : 4

   (b) Find the Routhian $\mathcal{R}(\theta, \dot{\theta}, \phi, p_\phi)$ (Actually, of course, there will be no dependence on ϕ).

   [Solution]

   Routh's procedure aims to combine the **Lagrangian formulation's conveniences for non-cyclic coordinates** with the **Hamiltonian formulation's advantages in handling cyclic coordinates**. Eventually, we want to construct a quantity known as **Routhian** as

$$\mathcal{R} = \mathcal{H}_{\text{cyclic}}(p_{s+1}, p_{s+2}...p_n; t) - \mathcal{L}_{\text{non-cyclic}}(q_1, q_2...q_s; \dot{q}_1, \dot{q}_2...\dot{q}_s; t) = \sum_{i=s+1}^{n} p_i\dot{q}_i - \mathcal{L} \tag{4}$$

where the cyclic coordinates are labelled as $q_{s+1}, q_{s+2}...q_n$. Then for the s non-cyclic coordinate, we can see that the Routhian automatically satisfies the Lagrange's eqution:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{R}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{R}}{\partial q_i} = 0 \tag{5}$$

and for the $n-s$ cyclic coordinates, the Routhian satisfies the Hamilton's equations of motion:

$$\begin{cases} \frac{\partial \mathcal{R}}{\partial q_i} = -\dot{p}_i = 0 \\ \frac{\partial \mathcal{R}}{\partial p_i} = \dot{q}_i \end{cases} \tag{6}$$

Hence, after a short review on the Routh's procedures (Note that the above derivations are **not required** to obtain full credit. I just put them here as a reference for the readers), we have the Routhian of this problem as

$$\begin{aligned} \mathcal{R} &= p_\phi \dot{q}_\phi - \mathcal{L} &(7)\\ &= ml^2 \sin^2\theta \dot{\phi} - \left(\frac{1}{2}ml^2\dot{\theta}^2 + \frac{1}{2}ml^2\sin^2\theta\dot{\phi}^2 - mgl(1-\cos\theta)\right) &(8)\\ &= \frac{1}{2}ml^2\sin^2\theta\dot{\phi}^2 - \frac{1}{2}ml^2\dot{\theta}^2 + mgl(1-\cos\theta) &(9)\\ &= \underbrace{\frac{p_\phi^2}{2ml^2\sin^2\theta}}_{\mathcal{H}_{\text{cyclic}}} - \underbrace{\left(\frac{1}{2}ml^2\dot{\theta}^2 - mgl(1-\cos\theta)\right)}_{\mathcal{L}_{\text{non-cyclic}}} &(10) \end{aligned}$$

- **(1 pt)** : Correct equation for evaluating the Routhian $\mathcal{R}$.
- **(1 pt)** : Correct final expression for the Routhian $\mathcal{R}$. Note that if students do not express $\mathcal{R}$ in terms of p_ϕ, only 0.5 credit will be awarded. Also note that if the students do not explicitly state the equation for evaluating $\mathcal{R}$, full credit should still be awarded if they got the correct final expression.

Total sub-points : 2

(c) Using $\mathcal{R}$ as the Lagrangian for the θ motion, show directly that the equation of motion can be written in the form

$$ml^2\frac{d^2\theta}{dt^2} = -\frac{d}{d\theta}V_{\text{eff}}(\theta), \tag{11}$$

and find the effective potential $V_{\text{eff}}(\theta)$, which will depend on the constant value of p_ϕ.

[Solution]

Note that θ is a non-cyclic coordinate, so the Routhian $\mathcal{R}$ satisfies the Lagrange's equation for θ as

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial \mathcal{R}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{R}}{\partial \theta} &= 0 &(12)\\ \frac{d}{dt}(-ml^2\dot{\theta}) &= \frac{\partial}{\partial\theta}\left(\frac{p_\phi^2}{2ml^2\sin^2\theta} + mgl(1-\cos\theta)\right) &(13)\\ ml^2\ddot{\theta} &= -\frac{\partial}{\partial\theta}\underbrace{\left(\frac{p_\phi^2}{2ml^2\sin^2\theta} + mgl(1-\cos\theta)\right)}_{V_{\text{eff}}} &(14)\\ ml^2\ddot{\theta} &= -\frac{\partial}{\partial\theta}V_{\text{eff}} = \frac{d}{d\theta}V_{\text{eff}} &(15) \end{aligned}$$

where the last line holds because V_{eff} is only a function of θ (Remember, p_ϕ is only a constant). The effective potential is given by

$$V_{\text{eff}} = \frac{p_\phi^2}{2ml^2\sin^2\theta} + mgl(1-\cos\theta) \tag{16}$$

- **(1 pt)** : Applying the Lagrange's equation to the Routhian.
- **(1 pt)** : Showing that $ml^2 \frac{d^2\theta}{dt^2} = -\frac{d}{d\theta} V_{\text{eff}}(\theta)$ with reasonable steps.
- **(1 pt)** : Correct expression for the effective potential V_{eff}.

Total sub-points : 3

(d) For motion in a vertical plane $p_\phi = 0$ we know there are oscillating solutions about $\theta = 0$ and running solutions where the pendulum passes through $\theta = \pi$ and continues to rotate in one direction. Show for $p_\phi \neq 0$ the θ motion is bounded $0 < \theta < \pi$ and is periodic.

[Solution]

Consider the effective potential:

$$V_{\text{eff}} = \frac{p_\phi^2}{2ml^2 \sin^2\theta} + mgl(1 - \cos\theta). \tag{17}$$

We note that V_{eff} will go to ∞ when $\theta \to 0$ or π. With a finite initial kinetic energy for ϕ, the particle can never reaches the limit for $\theta \to 0$ or π where the potential energy becomes infinity since energy is conserved!

Meanwhile, the particle can reach points near to $\theta = 0$ or π as long as the potential energy is finite. And, since energy is conserved, the kinetic energy for ϕ must be transformed to and from the effective potential periodically, making the motion a periodic one.

- **(1 pt)** : Arguing that since the effective potential blows up at $\theta = 0$ or π, the particle, with finite kinetic energy, can never reach these points.
- **(1 pt)** : Explaining why the particle can reach points **near to** the limits $\theta = 0$ or π.
- **(1 pt)** : Explaining why the resulting motion would be **periodic**.

Total sub-points : 3

(e) Show that conical motion $\theta = \theta_0 =$ constant, $p_\phi \neq 0$ only occurs for $\theta_0 < \pi/2$, i.e. for the string at an angle lower than horizontal.

[Solution]

For $\theta = \theta_0$=constant, we have $\ddot{\theta} = 0$ and hence

$$ml^2\ddot{\theta}|_{\theta=\theta_0} = -\left.\frac{\partial V_{\text{eff}}}{\partial\theta}\right|_{\theta=\theta_0} = 0 \tag{18}$$

$$\therefore \left.\frac{\partial V_{\text{eff}}}{\partial\theta}\right|_{\theta=\theta_0} = 0 \tag{19}$$

which is equivalent to finding the minimum point of the effective potential V_{eff}. Evaluating the partial derivative, we have

$$-\frac{p_\phi^2}{ml^2 \sin^3\theta_0} \cos\theta_0 + mgl \sin\theta_0 = 0 \tag{20}$$

$$\underbrace{\sin^4\theta_0}_{\geq 0} = \underbrace{\frac{p_p hi^2}{ml^3 g}}_{\geq 0} \cos\theta_0. \tag{21}$$

Note that the LHS is always larger than zero. To ensure the equality holds for any value of $p_p hi \neq 0$, we must have

$$\cos\theta_0 > 0. \tag{22}$$

Since $0 < \theta < \pi$, one must have

$$\theta_0 < \frac{\pi}{2} \tag{23}$$

Supplementary Box 7.1: You already knew this since you were a kid...
Even without learning anything about Lagrangian/Hamiltonian/Routhian, you should still be able to argue that $\theta_0 < \frac{\pi}{2}$ if the pendulum is rotating around the z-axis. Why?

Fetch yourself a computer mouse or a string tied to some mass (a pendulum). Hold the end of the string that is not tied to the mass, and start to move the mass around the vertical (your standing body is parallel to the vertical, in case you are not aware of this). Try to move the mass around in a way such that its height (and hence the angle θ) is not changing. You will see that no matter how hard you try, you can never make the mass circulate around the vertical at an angle larger or equal to $\theta = \frac{\pi}{2}$ (Not even at $\theta = \frac{\pi}{2}$)!

Why? Simply sketch a free-body diagram and convince yourself that it is impossible to have purely circular motion around the z-axis for $\theta \geq \frac{\pi}{2}$. Now you know why I said you knew the answer already since your were a kid!

- **(1 pt)** : Stating that $\frac{\partial V_{\text{eff}}}{\partial \theta} = 0$ when evaluted at $\theta = \theta_0$ =constant.
- **(1 pt)** : Correct equation relating $\sin\theta_0$ and $\cos\theta_0$.
- **(1 pt)** : Arguing that $\theta_0 < \frac{\pi}{2}$.

Total sub-points : 3

(f) Find the frequency ω of the small amplitude oscillations of θ about θ_0 for $p_\phi \neq 0$ in terms of g, l and θ_0, and the ratio $\omega/\dot{\phi}$ in terms of just θ_0, and find the limits for $\theta_0 \to 0$ and $\theta_0 \to \pi/2$. Reconcile your answer for $\theta_0 \to 0$ with what we expect for the small amplitude motion using x, y coordinates in the horizontal plane, namely periodic motion with frequency $\sqrt{g/l}$. A sketch of the motion in the x, y plane implied by the θ, ϕ motion may help. Give a physical description of the oscillations for $\theta_0 \to \pi/2$, including a simple argument for $\omega/\dot{\phi}$ for the motion in this limit.

[Solution]

For small oscillations, we look for solutions in the form

$$\theta = \theta_0 + \delta\theta \tag{24}$$

The equation of motion for θ gives

$$\begin{aligned}
ml^2\ddot{\delta\theta} &= -\frac{\partial V_{\text{eff}}}{\partial \theta}\bigg|_{\theta=\theta_0+\delta\theta} && (25)\\
&\approx \underbrace{-\frac{\partial V_{\text{eff}}}{\partial \theta}\bigg|_{\theta=\theta_0}}_{=0} - \frac{\partial^2 V_{\text{eff}}}{\partial \theta^2}\bigg|_{\theta=\theta_0}\delta\theta && (26)\\
&= -\frac{\partial}{\partial \theta}\left(-\frac{p_\phi^2}{ml^2\sin^3\theta}\cos\theta + mgl\sin\theta\right)\bigg|_{\theta=\theta_0}\delta\theta && (27)\\
&= -\left[-\frac{p_\phi^2}{ml^2}\left(\frac{-\sin^3\theta\sin\theta - \cos\theta\times 3\sin^2\theta\cos\theta}{\sin^6\theta}\right) + mgl\cos\theta\right]_{\theta=\theta_0}\delta\theta && (28)\\
&= -\left[\frac{p_\phi^2}{ml^2}\left(\frac{\sin^2\theta_0 + 3\cos^2\theta_0}{\sin^4\theta_0}\right) + mgl\cos\theta_0\right]\delta\theta && (29)\\
&= -\left[\frac{p_\phi^2}{ml^2}\left(\frac{1 + 2\cos^2\theta_0}{\sin^4\theta_0}\right) + mgl\cos\theta_0\right]\delta\theta && (30)
\end{aligned}$$

Using the result for $\frac{\partial V_{\text{eff}}}{\partial \theta}|_{\theta=\theta_0} = 0$, we get

$$p_\phi^2 = m^2 l^3 g\frac{\sin^4\theta_0}{\cos\theta_0}. \tag{31}$$

With this, we have

$$ml^2\ddot{\delta\theta} = -\left[\frac{m^2l^2g}{ml^2}\frac{\sin^4\theta_0}{\cos\theta_0}\left(\frac{1+2\cos^2\theta_0}{\sin^4\theta_0}\right) + mgl\cos\theta_0\right]\delta\theta \tag{32}$$

$$= -[mgl(\sec\theta_0 + 2\cos\theta_0) + mgl\cos\theta_0]\delta\theta \tag{33}$$

$$= -mgl(3\cos\theta_0 + \sec\theta_0)\delta\theta \tag{34}$$

Hence, we have

$$\ddot{\delta\theta} = -\frac{g}{l}(3\cos\theta_0 + \sec\theta_0)\delta\theta \tag{35}$$

$$\delta\theta = A\cos\left(\underbrace{\sqrt{\frac{g}{l}}\sqrt{3\cos\theta_0 + \sec\theta_0}}_{\omega}\, t + \Phi\right) \tag{36}$$

Remember that we have the conjugate momentum p_ϕ as

$$p_\phi = ml^2\sin^2\theta\dot{\phi}, \tag{37}$$

and so we can write

$$\dot{\phi} = \frac{p_\phi}{ml^2\sin^2\theta} \tag{38}$$

Therefore, we have the required ratio as

$$\frac{\omega}{\dot{\phi}} = \left(\sqrt{\frac{g}{l}}\sqrt{3\cos\theta_0 + \sec\theta_0}\right) \div \left(\frac{p_\phi}{ml^2\sin^2\theta_0}\right) \tag{39}$$

$$= \left(\sqrt{\frac{g}{l}}\sqrt{3\cos\theta_0 + \sec\theta_0}\right) \div \left(\sqrt{\frac{g}{l}}\frac{1}{\sqrt{\cos\theta_0}}\right) \tag{40}$$

$$= \sqrt{3\cos^2\theta_0 + 1} \tag{41}$$

(1) For the limit $\theta_0 \to 0$, we have

$$\omega \to 2\sqrt{\frac{g}{l}} \tag{42}$$

and hence

$$\theta = A\cos\left(2\sqrt{\frac{g}{l}}t + \Phi\right) + \Theta \tag{43}$$

and

$$\frac{\omega}{\dot{\phi}} \to 2 \tag{44}$$

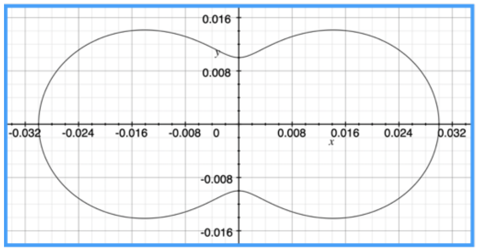
$$\dot{\phi} \to \frac{\omega}{2} = \sqrt{\frac{g}{l}} \tag{45}$$

$$\phi = \sqrt{\frac{g}{l}}t + \phi_0. \tag{46}$$

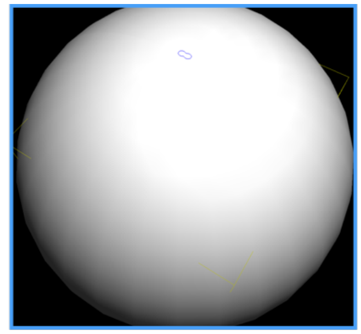
where Θ is a very small quantity (Remember, θ can never be zero, we are just pushing it to the limit). Then, we have

$$\begin{cases} x = l\sin\theta\cos\phi = l\sin\left[A\cos\left(2\sqrt{\frac{g}{l}}t + \Phi\right) + \Theta\right]\cos\left(\sqrt{\frac{g}{l}}t + \phi_0\right) \\ y = -l\sin\theta\sin\phi = -l\sin\left[A\cos\left(2\sqrt{\frac{g}{l}}t + \Phi + \Theta\right)\right]\sin\left(\sqrt{\frac{g}{l}}t + \phi_0\right) \end{cases} \tag{47}$$

With these parametric equations, we can draw graphs as shown below:



Parametric curves projected on the x-y plane



3D Parametric curves on the sphere with radius l

This shows that small oscillations about $\theta = 0$ gives an oscillatory motion with $\omega_\theta = 2\omega_\phi = 2\sqrt{\frac{g}{l}}$. When θ_0 is very small, the trajectory actually just changes a little bit from a circular orbit, and becomes a slightly "elliptical orbit".

(2) For the limit $\theta_0 \to \frac{\pi}{2}$, we have

$$\omega \to \sqrt{\frac{g}{l}}\sqrt{0 + \frac{1}{0}} \to \infty \tag{48}$$

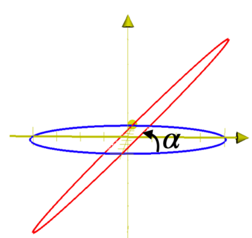
and

$$\frac{\omega}{\dot{\phi}} \to 1 \tag{49}$$

$$\dot{\phi} \to \omega = \infty. \tag{50}$$

What is happening here? The thing is, we are forcing the pendulum to reach an unreachable limit $\to$ a horizontal plane! Why is this unreachable? For a pendulum to move around only in the horizontal plane $\theta = \frac{\pi}{2}$, there is **NO VERTICAL COMPONENT** from the tension of the rope that can balance the weight acting on the pendulum, and hence this is never an attainable limit! For this to "happen", the pendulum must be circulating at an **infinite speed**! In other words, such motion is a planar motion. If we actually tilt the horizontal plane by a small angle, say $\delta\theta = \alpha$, then the pendulum will also oscillate about $\theta = \frac{\pi}{2}$, with

$$\omega_\theta = \omega_\phi \tag{51}$$



- **(2 pt)** : Solving for the small oscillation frequency ω.
- **(1 pt)** : Correct expression for the small oscillation frequency ω.
- **(1 pt)** : Correct expression for the ratio $\frac{\omega}{\dot{\phi}}$.
- **(1.5 pt)** : Correct expressions for θ, $\frac{\omega}{\dot{\phi}}$ and ϕ for the limit $\theta \to 0$.
- **(2 pt)** : Correct expression for the corresponding x and y with the limit $\theta \to 0$.
- **(2 pt)** : Reasonable discussion on how the trajectories for $\theta \to 0$ look like - Students should mention that it is a periodic motion with $\omega_\theta = 2\omega_\phi$, and also the trajectory is no longer a circular orbit.

- **(1.5 pt)** : Correct expressions for θ, $\frac{\omega}{\dot\phi}$ and ϕ for the limit $\theta \to \frac{\pi}{2}$.
- **(1 pt)** : Explain why both ω and $\dot\phi$ go to ∞ in the case of $\theta \to \frac{\pi}{2}$.
- **(1 pt)** : Explain what will happen if we tilt the planar motion by an angle α.

Total sub-points : 13

2. **Poisson Brackets with angular momentum:** (H&F problem 6.15)

(a) Calculate all the Poisson brackets of the components of $\vec{r}$ and $\vec{p}$ with each other (for example, $[x, y]$, $[x, p_x]$, $[x, p_y]$, etc.).

[Solution]

Poisson bracket is defined as

$$[A, B] = \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i} \tag{52}$$

We use q_i to represent x,y and z and p_i to represent the corresponding momenta.

First consider

$$[q_i, q_j] = \underbrace{\frac{\partial q_i}{\partial q_m}}_{\delta_{im}}\underbrace{\frac{\partial q_j}{\partial p_m}}_{0} - \underbrace{\frac{\partial q_i}{\partial p_m}}_{0}\underbrace{\frac{\partial q_j}{\partial q_m}}_{\delta_{jm}} = 0 \tag{53}$$

where we have used the fact that q_i and p_j are independent coordinates.

Similarly, we have

$$[p_i, p_j] = \underbrace{\frac{\partial p_i}{\partial q_m}}_{0}\underbrace{\frac{\partial p_j}{\partial p_m}}_{\delta_{jm}} - \underbrace{\frac{\partial p_i}{\partial p_m}}_{\delta_{im}}\underbrace{\frac{\partial p_j}{\partial q_m}}_{\delta_0} = 0 \tag{54}$$

Finally, we have

$$[q_i, p_j] = \underbrace{\frac{\partial q_i}{\partial q_m}}_{\delta_{im}}\underbrace{\frac{\partial p_j}{\partial p_m}}_{\delta_{jm}} - \underbrace{\frac{\partial q_i}{\partial p_m}}_{0}\underbrace{\frac{\partial p_j}{\partial q_m}}_{\delta_0} = \delta_{im}\delta_{jm} = \delta_{ij} \tag{55}$$

Supplementary Box 7.2 - Poisson brackets for canonical coordinates

Before we begin, let's have a small discussion on what are canonical coordinates. In Hamiltonian dynamics, canonical coordinates q_i, p_i must satisfy the Hamilton's equations of motion,

$$\begin{cases} \dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} \\ \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i} \end{cases} \tag{56}$$

where q_i and p_i are the generalized coordinates and momenta respectively.

In general, canonical coordinates satisfy the following Poisson bracket properties:

(a) $[q_i, q_j] = 0$.

(b) $[p_i, p_j] = 0$.

(c) $[q_i, p_j] = \delta_{ij}$.

It is not always the case that we are using the "best" coordinate system. For instance, in 2D problems we can use Cartesian coordinates and polar coordinates. If we want to transform from one system to another, we will need to use something called **canonical transformation**. Read Goldstein Chapter 9 to find out more.

- **(1 pt)** : Correct result for $[q_i, q_j]$.

- **(1 pt)** : Correct result for $[p_i, p_j]$.
- **(2 pt)** : Correct result for $[q_i, p_j]$.

Total sub-points : 4

(b) Calculate all the Poisson brackets of the components of $\vec{r}$ and $\vec{p}$ with the components of the angular momentum $\vec{l} = \vec{r} \times \vec{p}$ (for example, $[x, l_x]$, $[x, l_y]$, $[p_x, l_x]$, etc.). You may find it convenient to use the Levi-Civita completely anti-symmetric tensor ϵ_{ijk} (look it up) such that $l_i = \epsilon_{ijk} r_j p_k$.

[Solution]

The Levi-Civita symbol is defined as

$$\epsilon_{ijk} = \begin{cases} 1, & \text{if } i,j \text{ and } k \text{ are cyclic}(123, 231, 312) \\ -1, & \text{if } i,j \text{ and } k \text{ are acyclic}(321, 213, 132) \\ 0, & \text{otherwise} \end{cases} \tag{57}$$

Since

$$\vec{l} = \vec{r} \times \vec{p} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ r_x & r_y & r_z \\ p_x & p_y & p_z \end{vmatrix} \tag{58}$$

$$= \underbrace{\hat{x}(r_y p_z - r_z p_y)}_{=r_i p_j \epsilon_{ijx} \hat{x}} - \underbrace{\hat{y}(r_x p_z - r_z p_x)}_{r_i p_j \epsilon_{ijy} \hat{y}} + \underbrace{\hat{z}(r_x p_y - r_y p_x)}_{r_i p_j \epsilon_{ijz} \hat{z}}, \tag{59}$$

we notice that

$$l_k = r_i p_j \epsilon_{ijk} \hat{e}_k \tag{60}$$

where $\hat{e}_k$ is the unit vector along the k direction.
With such, we have

$$[q_i, l_j] = [q_i, q_a p_b \epsilon_{abj}] \tag{61}$$

$$= \underbrace{\frac{\partial q_i}{\partial q_m}}_{\delta_{im}} \frac{\partial (q_a p_b \epsilon_{abj})}{\partial p_m} - \frac{\partial (q_a p_b \epsilon_{abj})}{\partial q_m} \underbrace{\frac{\partial q_i}{\partial p_m}}_{=0} \tag{62}$$

$$= \frac{\partial (q_a p_b \epsilon_{abj})}{\partial p_i} \tag{63}$$

$$= q_a \epsilon_{abj} \underbrace{\frac{\partial p_b}{\partial p_i}}_{\delta_{bi}} \tag{64}$$

$$= q_a \epsilon_{aij} \tag{65}$$

$$= \begin{cases} 0, & \text{if } i = j \\ +q_b, & \text{if } bij \text{ are different and cyclic} \\ -q_b, & \text{if } bij \text{ are different and acyclic} \end{cases} \tag{66}$$

Hence, we have

$$\begin{array}{ccc} [x, l_x] = 0 & [x, l_y] = z & [x, l_z] = -y \\ [y, l_x] = -z & [y, l_y] = 0 & [y, l_z] = x \\ [z, l_x] = y & [z, l_y] = -x & [z, l_z] = 0 \end{array} \tag{67}$$

Similarly, we have

$$
\begin{aligned}
[p_i, l_j] &= [p_i, q_a p_b \epsilon_{abj}] && (68)\\
&= \underbrace{\frac{\partial p_i}{\partial q_m}}_{=0} \frac{\partial (q_a p_b \epsilon_{abj})}{\partial p_m} - \frac{\partial (q_a p_b \epsilon_{abj})}{\partial q_m} \underbrace{\frac{\partial p_i}{\partial p_m}}_{\delta_{im}} && (69)\\
&= -\frac{\partial (q_a p_b \epsilon_{abj})}{\partial q_i} && (70)\\
&= -\delta_{ai} p_b \epsilon_{abj} && (71)\\
&= -p_b \epsilon_{ibj} && (72)\\
&= p_b \epsilon_{ijb} && (73)\\
&= \begin{cases} 0, & \text{if } i = j \\ +p_b, & \text{if } ijb \text{ are different and cyclic} \\ -p_b, & \text{if } ijb \text{ are different and acyclic} \end{cases} && (74)
\end{aligned}
$$

where in the second last step we have interchanged two indicies in the Levi-Civita tensor. Because of the anti-symmetric property, the negative sign is gone.
Hence, we have

$$
\begin{array}{ccc}
[p_x, l_x] = 0 & [p_x, l_y] = p_z & [p_x, l_z] = -p_y \\
[p_y, l_x] = -p_z & [p_y, l_y] = 0 & [p_y, l_z] = p_x \\
[p_z, l_x] = p_y & [p_z, l_y] = -p_x & [p_z, l_z] = 0
\end{array} \qquad (75)
$$

- **(2 pt)** : Correct result for $[q_i, l_i]$.
- **(2 pt)** : Correct result for $[p_i, l_i]$

Total sub-points : 4

(c) Prove that $[l_x, l_y] = l_z$ for all cyclic permutations of l_x, l_y, l_z.
[Solution]
We have

$$
\begin{aligned}
[l_i, l_j] &= [q_a p_b \epsilon_{abi}, q_c p_d \epsilon cdj] && (76)\\
&= \frac{\partial (q_a p_b \epsilon_{abi})}{\partial q_m} \frac{\partial (q_c p_d \epsilon_{cdj})}{\partial p_m} - \frac{\partial (q_c p_d \epsilon_{cdj})}{\partial q_m} \frac{\partial (q_a p_b \epsilon_{abi})}{\partial p_m} && (77)\\
&= (\delta_{am} p_b \epsilon_{abi})(q_c \delta_{dm} \epsilon_{cdj}) - (q_a \delta_{bm} \epsilon_{abi})(\delta_{cm} p_d \epsilon_{cdj}) && (78)\\
&= p_b q_c \epsilon_{abi} \epsilon_{caj} - q_a p_d \epsilon_{abi} \epsilon_{bdj} && (79)
\end{aligned}
$$

For product of the Levi-Civita tensor, we have the following property (see Supplementary Box 7.3 to learn more):

$$\epsilon_{aij} \epsilon_{amn} = \delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}. \qquad (80)$$

With this, we have

$$
\begin{aligned}
[l_i, l_j] &= p_b q_c (\delta_{bj} \delta_{ic} - \delta_{bc} \delta_{ij}) - q_a p_d (\delta_{id} \delta_{aj} - \delta_{ij} \delta_{ad}) && (81)\\
&= p_j q_i - p_c q_c \delta_{ij} - q_j p_i + \underbrace{q_a p_c \delta_{ij}}_{= q_c p_c \delta_{ij}} && (82)\\
&= q_i p_j - p_i q_j && (83)\\
&= l_k && (84)
\end{aligned}
$$

where in the second line we replaced a dummy variable c in a term by another dummy variable a.

Supplementary Box 7.3: More about the Levi-Civita tensor

When two Levi-Civita tensors are multiplied together, they can acutally be written as a determinant:

$$\epsilon_{abc}\epsilon_{ijk} = \begin{vmatrix} \delta_{ai} & \delta_{aj} & \delta_{ak} \\ \delta_{bi} & \delta_{bj} & \delta_{bk} \\ \delta_{ci} & \delta_{cj} & \delta_{ck} \end{vmatrix} \tag{85}$$

If one of the indices is repeated (say $a = i$) in the two tensors, summation is implied. Since we are only dealing with three dimensional objects, there is only one possible case such that $a \neq b \neq c$ and the tensor will not vanish. Consider this case, we have

$$\epsilon_{abc}\epsilon_{ajk} = \begin{vmatrix} \delta_{aa} & \delta_{aj} & \delta_{ak} \\ \delta_{ba} & \delta_{bj} & \delta_{bk} \\ \delta_{ca} & \delta_{cj} & \delta_{ck} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \delta_{bj} & \delta_{bk} \\ 0 & \delta_{cj} & \delta_{ck} \end{vmatrix} = \delta_{bj}\delta_{ck} - \delta_{bk}\delta_{cj} \tag{86}$$

and that's how our property pops out! =)

- **(4 pt)** : Correct result for $[l_i, l_j]$. Partial credit should be given for reasonable steps but incorrect results.

Total sub-points : 4

3. **Poisson Brackets with constants of motion**: (H&F problem 6.14).

   Consider the uniform motion of a free particle of mass m. The Hamiltonian is a constant of the motion, and so is the quantity $F(x, p, t) \equiv x - pt/m$.

   (a) Compare $[H, F]$ with $\frac{\partial F}{\partial t}$. Prove that F is also a constant of the motion.

   [Solution]

   First, we have

$$\frac{\partial F}{\partial t} = -\frac{p}{m}. \tag{87}$$

   Now, the Hamiltonian is given by

$$\mathcal{H} = \frac{p^2}{2m}. \tag{88}$$

   where $p = m\dot{x}$.

   We have

$$\begin{aligned} [\mathcal{H}, F] &= \frac{\partial \mathcal{H}}{\partial x}\frac{\partial F}{\partial p} - \frac{\partial F}{\partial x}\frac{\partial \mathcal{H}}{\partial p} && (89) \\ &= 0 - \frac{p}{m} && (90) \\ &= -\frac{p}{m}. && (91) \end{aligned}$$

   Therefore, we see that

$$[\mathcal{H}, F] = \frac{\partial F}{\partial t}. \tag{92}$$

   To prove that F is a constant of motion, note that

$$\frac{dF}{dt} = \underbrace{[F, \mathcal{H}]}_{=-[\mathcal{H}, F]} + \frac{\partial F}{\partial t} = -\frac{\partial F}{\partial t} + \frac{\partial F}{\partial t} = 0 \tag{93}$$

   Hence, we can immediately conclude that F is also a constant of motion.

   - **(1 pt)** : Evaluating $\frac{\partial F}{\partial t}$.

- **(1 pt)** : Evaluating $[\mathcal{H}, F]$.
- **(1 pt)** : Stating that $[\mathcal{H}, F] = \frac{\partial F}{\partial t}$.
- **(1 pt)** : Arguing that F is also a constant of motion with reasonable explanation.

Total sub-points : 4

(b) Prove that the Poisson bracket of two constants of motion is itself a constant of the motion, even if the constants $F(x,p,t)$ and $G(x,p,t)$ depend explicitly on the time. (Part a is one example of this).

[Solution]

The Poisson bracket for two constants of motion F and G is

$$[F,G] = \frac{\partial F}{\partial x}\frac{\partial G}{\partial p} - \frac{\partial G}{\partial x}\frac{\partial F}{\partial p} = R. \tag{94}$$

We consider

$$\frac{dR}{dt} = [R,\mathcal{H}] + \frac{\partial R}{\partial t} \tag{95}$$

For the second term, we have

$$\frac{\partial R}{\partial t} = \frac{\partial}{\partial t}[F,G] = \left[\frac{\partial F}{\partial t}, G\right] + \left[F, \frac{\partial G}{\partial t}\right] \tag{96}$$

Recall that for any constant of motion A, it satisfies

$$0 = \frac{dA}{dt} = [A,\mathcal{H}] + \frac{\partial A}{\partial t} \tag{97}$$

$$[A,\mathcal{H}] = -\frac{\partial A}{\partial t}. \tag{98}$$

We will need one more ingredient to complete the proof - The Jacobi identity (See Supplementary Box 7.4 to learn more about this identity):

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0 \tag{99}$$

With these, the first term in $\frac{dR}{dt}$ can be written as

$$\begin{aligned}
[R,\mathcal{H}] &= [[F,G],\mathcal{H}] && (100)\\
&= -[\mathcal{H},[F,G]] && (101)\\
&= [F,[G,\mathcal{H}]] + [G,[\mathcal{H},F]] && (102)\\
&= \left[F, -\frac{\partial G}{\partial t}\right] + \left[G, \frac{\partial F}{\partial t}\right] && (103)\\
&= \left[\frac{\partial G}{\partial t}, F\right] - \left[\frac{\partial F}{\partial t}, G\right] && (104)\\
&= -\left[F, \frac{\partial G}{\partial t}\right] - \left[\frac{\partial F}{\partial t}, G\right] && (105)
\end{aligned}$$

and hence

$$\frac{dR}{dt} = -\left[F,\frac{\partial G}{\partial t}\right] - \left[\frac{\partial F}{\partial t}, G\right] + \frac{\partial}{\partial t}[F,G] = \left[\frac{\partial F}{\partial t}, G\right] + \left[F, \frac{\partial G}{\partial t}\right] = 0 \tag{106}$$

Therefore, we have the Poisson bracket of two constants of motion is itself a constant of motion.

Supplementary Box 7.3: The Jacobi Identity and Leibniz's rule
The Jacobi identity states that

$$[A,[B,C]]+[B,[C,A]]+[C,[A,B]]=0 \tag{107}$$

To prove this, readers can write out all the components in each of the Poisson brackets, and show that they will eventually cancel out each other. We leave it as an exercise for keen readers. In any case, this is not a hard thing to prove.
There is another potentially useful identity that readers may be interested - The Leibniz's rule:

$$[AB,C]=[A,C]B+A[B,C] \tag{108}$$

Take these with you! You might need to use them again in the future, or you might actually encounter similar things again in Quantum Mechanics.

- **(1 pt)** : Considering the total time derivative of R.
- **(1 pt)** : Correct expression for the partial time derivative of R.
- **(2 pt)** : Correct final expression for $[R,\mathcal{H}]$.
- **(2 pt)** : Show that R is also a constant of motion. Note that students may not follow the suggested solutions here. Judging by a case by case basis, full credit should still be given if their approach is flawless and correct.

Total sub-points : 6

(c) Show in general that if the Hamiltonian and a quantity F are constants of the motion then $\frac{\partial F}{\partial t}$ is a constant of motion also.
[Solution]
Since F is a constant of motion, we have

$$\frac{\partial F}{\partial t}=-[F,\mathcal{H}] \tag{109}$$

Now we consider

$$\frac{d}{dt}\left(\frac{\partial F}{\partial t}\right)=-\frac{d}{dt}[F,\mathcal{H}] \tag{110}$$

$$\left[\frac{\partial F}{\partial t},\mathcal{H}\right]+\frac{\partial^2 F}{\partial t^2}=\left[\frac{dF}{dt},\mathcal{H}\right]+\left[F,\frac{d\mathcal{H}}{dt}\right]=0 \tag{111}$$

$$\frac{\partial^2 F}{\partial t^2}=-\left[\frac{\partial F}{\partial t},\mathcal{H}\right] \tag{112}$$

where we have used the fact that $\frac{dF}{dt}=\frac{d\mathcal{H}}{dt}=0$ since they are constants of motion.
Hence, we see that $\frac{\partial F}{\partial t}$ is also a constant of motion.

Supplementary Box 7.4: What about the n^{th} partial time derivative?
Here, I challenge readers to prove that in general, if F and H are both constants of motion, then the n^{th} partial time derivatives of F is also a constant of motion. You may find the method of **mathematical induction** useful.

- **(1 pt)** : Calculating the total time derivative of $\frac{\partial F}{\partial t}$.
- **(1 pt)** : Using the fact that H and F are constants of motion.
- **(2 pt)** : Deriving the final expression in the solution.

Total sub-points : 4

4. **Light, a bullet and a ruler** In frame S, a ray of light and the path of a bullet with speed u run along the edge of a stationary ruler at an angle θ to the x-axis in the xy plane. Now look at the angles θ' relative to the x'-axis observed in the frame S' moving with speed v along the x-axis of S.

   (a) What is the angle θ'_r that the ruler is measured to have?

   [Solution]

   Readers are reminded that without loss of generality, we are using $c = 1$ throughout the rest of this problem set.

   Since frame S' is only moving along the x-axis of the S frame, relativity will only affect measurements along the x-direction. Let L be the **proper length** of the ruler (by proper length we mean the length measured by an observing moving together with the ruler (i.e. a stationary observer)). The y-component of the length of the ruler is agreed to be $L\sin\theta$ by both observers in frame S and S'. The only difference comes from the x-component, where in the S' frame it is measured to be

$$L'_x = \frac{L_x}{\gamma} = L\cos\theta\sqrt{1-v^2} \tag{113}$$

   Hence, the angle of the ruler is measured to be

$$\tan\theta'_r = \frac{L\sin\theta}{L\cos\theta\sqrt{1-v^2}} = \gamma\tan\theta = \frac{\tan\theta}{\sqrt{1-v^2}} \tag{114}$$

   - **(1 pt)** : Correct x-component of length of ruler in the S' frame.
   - **(1 pt)** : Correct expression for the angle θ'_r

   Total sub-points : 2

   (b) What is the angle θ'_b of the path of the bullet?

   [Solution]

   From Lorentz transformation,

$$\begin{bmatrix} t' \\ x' \\ y' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma v & 0 \\ -\gamma v & \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ x \\ y \end{bmatrix}. \tag{115}$$

   We have

$$\begin{cases} t' = \gamma t - \gamma v x \\ x' = -\gamma v t + \gamma x \end{cases} \quad \rightarrow \quad \begin{cases} dt' = \gamma dt - \gamma v dx \\ dx' = -\gamma v dt + \gamma dx \end{cases} \tag{116}$$

   Hence,

$$\frac{dx'}{dt'} = \frac{-\gamma v dt + \gamma dx}{\gamma dt - \gamma v dx} = \frac{-v + \frac{dx}{dt}}{1 - v\frac{dx}{dt}} = \frac{u_x - v}{1 - u_x v} \tag{117}$$

   and

$$\frac{dy'}{dt'} = \frac{dy}{\gamma dt - \gamma v dx} = \frac{\frac{dy}{dt}}{\gamma(1 - v\frac{dx}{dt})} = \frac{u_y}{\gamma(1 - u_x v)} \tag{118}$$

   where u_x and u_y are the x and y component of the velocity of the bullet in the S frame respectively. Finally, we have

$$\tan\theta'_b = \frac{dy'}{dx'} = \frac{u_y}{\gamma(1 - u_x v} \frac{1 - u_x v}{u_x - v} = \frac{u_y}{\gamma(u_x - v)} = \frac{u\sin\theta}{\gamma(u\cos\theta - v)} \tag{119}$$

- **(2 pt)** : Finding dx' and dy'.
- **(2 pt)** : Correct final expression for $\tan\theta_b'$.

Total sub-points : 4

(c) What is the angle θ_l' of the light ray?

[Solution]

Readers might be tempted to use the invariance statement for light speed and claim that the angle will be the same as θ. This is **NOT** the case because when the light speed is projected to the x-axis, it is no longer equal to the light speed and hence the argument will fail.

We can find the answer easily, however, by substituting u by 1 (remember, we set $c = 1$) in the expression we obtained in the last part. Hence the required angle is simply

$$\tan\theta_l' = \frac{\sin\theta}{\gamma(\cos\theta - v)} \tag{120}$$

- **(1 pt)** : Substituting $u = 1$ into the expression obtained in the last part.
- **(1 pt)** : Correct expression for the angle θ_l'.

Total sub-points : 2

(d) Are the three angles θ_r', θ_b', θ_l' equal? Do the bullet and light ray run along the edge of the ruler in the S' frame? Explain your answer in terms of frame independent events.

[Solution]

Obviously, the three angles θ_r', θ_b', θ_l' are not equal. However, this does not mean that the light ray and bullet do not run along the edge of the ruler in the S" frame.

The paths of the light ray and bullet define a series of events in spacetime: (A) The bullet and light ray start moving from the origin (one end of the ruler); (B) The light ray reaches the other end of the ruler; (C) The bullet reaches the other end of the ruler. Regardless of the frame we are in, these events always happen, i.e. they are frame independent events. Observers in different frame may not agree on the **order** that these events occur, but they will always agree that these events did actually happen!

Here let me give you a more intuitive example: Consider a train moving along a stationary train track along the $y = x$ line in the S frame, which is stationary. Consider another person who is running along the x-axis. If we say that in that observer's frame the train does not move along the train track, then a tragedy would have already happened in the moving observer's frame, while no accident is occuring in the stationary observer's frame. This will not make any sense - Will the train passengers be injured or not? Obviously, if the train is moving along the track in one frame, it must move along the track in all other frames! The observers may argue about the chronology of the events that happen (does the train stop by Pasadena first or stop by San Francisco first), or the place where the events happen, but not the occurrence of these events.

- **(1 pt)** : Stating that the three angles are different.
- **(1 pt)** : Stating that the bullet and the light ray still move along the edge of the ruler in the S' frame.
- **(1 pt)** : Arguing why the bullet and the light ray still move along the edge of the ruler in the S' frame.

Total sub-points : 3

5. **Addition of velocities:** A particle has velocity $\vec{u} = (u_x', u_y', u_z')$ in the frame S' moving at speed v along the x-axis relative to frame S (our standard configuration). Derive yourself using $u_x' = dx'/dt'$

and $u_x = dx/dt$ etc. and the Lorentz transformations for dx, dt (but you need not hand this part in) or find in a textbook, the expressions for the components of the velocity $\vec{u}$ in S

$$u_x = \frac{u'_x + v}{1 + u'_x v}, \qquad u_y = \frac{u'_y}{\gamma_v(1 + u'_x v)}, \qquad u_z = \frac{u'_z}{\gamma_v(1 + u'_x v)} \tag{121}$$

with $\gamma_v = (1 - v^2)^{-1/2}$.

A stick is measured to be length L in its rest frame S. In a frame S' moving at speed $\frac{3}{5}$ along the $+x$ axis of S the stick is seen to move by at a speed $\frac{3}{5}$ along the $-x'$ direction. In the frame S' a ball also moves by at the same speed but in the $+x'$ direction.

(a) In the frame S', I calculate the time $\Delta t'$ it takes the ball to pass the stick as the contracted length L/γ with $\gamma = \frac{5}{4}$ divided by the relative speed $\frac{6}{5}$. Is this the correct answer? Is it consistent with the idea that "nothing travels faster than the speed of light"?

[Solution]

This is indeed the **correct answer**. Remember, velocities are frame dependent, but when it comes to "relative" velocities **in the same frame**, they can be anything up to $2c$. The statement "nothing travels faster than the speed of light" applies to only velocities measured relative to a stationary observer in a particular frame.

- **(1 pt)** : Stating that the answer is correct.
- **(1 pt)** : Arguing that the answer is consistent with the idea that "nothing travels faster than the speed of light".

Total sub-points : 2

(b) Find the speed of the ball in the rest frame of the stick, and the speed of the stick in the rest frame of the ball. What is the length of the stick measured in the frame S'' moving with the ball?

[Solution]

We have the speed of the ball measured in the rest frame of the stick as

$$u_{\text{ball}} = \frac{\frac{3}{5} + \frac{3}{5}}{1 + \frac{3}{5} \times \frac{3}{5}} = \frac{\frac{6}{5}}{1 + \frac{9}{25}} = \frac{15}{17} \tag{122}$$

In the S'' frame, S frame (and hence the stick) is moving at a speed of

$$u''_S = u''_{\text{stick}} = \frac{(-\frac{3}{5}) + (-\frac{3}{5})}{1 + (-\frac{3}{5})(-\frac{3}{5})} = \frac{-\frac{6}{5}}{1 + \frac{9}{25}} = -\frac{15}{17} \tag{123}$$

The length of the rod in the S'' frame is hence given by

$$L'' = \frac{L}{\gamma_{S''}} = L\sqrt{1 - \left(-\frac{15}{17}\right)^2} = \frac{8}{17}L. \tag{124}$$

- **(1 pt)** : Correct speed of ball in the rest frame of the stick.
- **(1 pt)** : Correct speed of stick in the rest frame of the ball.
- **(1 pt)** : Correct length of rod as measured in the rest frame of the ball.

Total sub-points : 3

(c) From part (b) find the time $\Delta t''$ measured in the frame of reference moving with the ball between the two ends of the stick passing the ball. Show that this is consistent with the time measured in S' evaluated in part (a) and the Lorentz transformation between the two frames S' and S'' of the coordinates for the two events: ball coincides with one end of stick; ball coincides with other end of stick.

[Solution]
The time measured in the S'' frame is simply given by

$$\Delta t'' = \frac{L''}{|u_{S''}|} = \frac{\frac{8}{17}L}{\frac{15}{17}} = \frac{8}{15}L \tag{125}$$

To transform back from the S'' frame to the S' frame, recall that Lorentz transformation dictates that

$$\begin{bmatrix} t' \\ x' \end{bmatrix} = \begin{bmatrix} \gamma & \gamma v \\ \gamma v & \gamma \end{bmatrix} \begin{bmatrix} t'' \\ x'' \end{bmatrix} \tag{126}$$

Note that the minus sign in the Lorentz transformation disappears since we are doing an inverse Lorentz transformation (another way to look at this is that frame S' is moving to the left of S'' frame).
We have

$$t' = \gamma t'' + \gamma v x'' \tag{127}$$

$$\Delta t' = \gamma \Delta t'' + \gamma v \Delta x'' \tag{128}$$

Note that $\Delta x'' = 0$ since the stick passes through the ball at exactly the **same location** in the rest frame of the ball! The frame S' is moving at a speed $\frac{3}{5}$ to the left of frame S'', so we have

$$\Delta t' = \gamma \Delta t'' = \frac{1}{\sqrt{1 - (-\frac{3}{5})^2}} \times \frac{8}{15}L = \frac{2}{3}L \tag{129}$$

which is exactly the same as the answer obtained in part (a)!

- **(1 pt)** : Correct result for $\Delta t''$.
- **(1 pt)** : Deriving the Lorentz transformation between $\Delta t'$ and $\Delta t''$.
- **(1 pt)** : Stating that $\Delta x'' = 0$ in the S'' frame.
- **(1 pt)** : Showing that $\Delta t'$ is the same as the answer obtained in part (a).

Total sub-points : 4

Maximum points obtainable for this problem set: 73 pt