Physics 106a: Assignment 6 **(Solutions prepared by Yanlong Shi)**

15 November, 2019; due 7pm Friday, 22 November in the "Ph106 In Box" in East Bridge mailbox.

Scattering, and Hamiltonian Formulation

Reading

Scattering: Hand and Finch Chapter 4, especially 4.7 and problems 28-30.
Hamiltonian mechanics: Hand and Finch Chapter 5.

Problems

1. **Cross section for Rutherford scattering** (H& F problem 4.28):

   We sketched out the derivation of the Rutherford scattering cross section in class; here, you will fill out all the steps in detail. (Other textbooks go through the calculation in detail, and it's ok if you had read about it earlier). A parallel beam of energetic alpha particles (helium nuclei from radium decay) of kinetic energy E is sent towards a thin gold foil, scattering off of individual gold nuclei.

   (a) Assume that the potential is that of a "point-like" scatterer, so that $V(r) = Z_{Au}Z_\alpha e^2/r$ down to the smallest values of r accessible by the experimental conditions. Starting with the formula relating the impact parameter b to the scattering angle θ, derive the differential cross section for Rutherford scattering:

   $$\frac{d\sigma}{d\Omega} = \left(\frac{Z_{Au}Z_\alpha e^2}{4E}\right)^2 \frac{1}{\sin^4\frac{\theta}{2}}$$

   You can assume that the gold nucleus target is so much heavier than the beam alpha particle that it is at rest in the center of mass (neglect recoil), and the total center-of-mass energy is equal to the kinetic energy of the beam alpha particle, E.

   [Solution]

   Starting from the two-body scattering, we fist find the equation of motion by

   $$\begin{aligned}\mu\ddot{r} &= -\frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}r} \\ &= -\frac{l^2}{\mu r^3} - \frac{k}{r^2}\end{aligned} \tag{1}$$

   Here we have set $k = Z_{Au}Z_\alpha e^2$. By substitute $u = 1/r$, and note that

   $$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}\phi}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}\phi} = \frac{l}{\mu r^2}\frac{\mathrm{d}}{\mathrm{d}\phi}; \tag{2}$$

   $$\ddot{r} = \frac{l}{\mu r^2}\frac{\mathrm{d}}{\mathrm{d}\phi}\left(\frac{l}{\mu r^2}\frac{\mathrm{d}r}{\mathrm{d}\phi}\right) = -\frac{l^2u^2}{\mu^2}\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2}, \tag{3}$$

   we may transfer the equation of motion to be

   $$\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} + u = -\frac{\mu k}{l^2}. \tag{4}$$

   The solution to the ODE is

   $$u = \frac{1}{r} = \frac{\epsilon\cos\phi}{p} - \frac{1}{p}. \tag{5}$$

   In this expression, we have set $p = l^2/(\mu k)$. The eccentricity ϵ is to be determined.

Let $r \to \infty$, the asymptotes are solution to $\epsilon \cos\phi - 1 = 0$. The two neighboring solutions are $\phi_\pm = \pm\phi_\infty = \pm \arccos(1/\epsilon)$, and the scattering angle θ is $\pi - 2\phi_\infty$, thus

$$\cos\theta = -\cos(2\phi_\infty) = -2\cos^2\phi_\infty + 1 = -\frac{2}{\epsilon^2} + 1; \tag{6}$$

$$\epsilon^2 = \frac{1}{\sin^2(\theta/2)}. \tag{7}$$

We also have the radial velocity to be

$$\dot{r} = \frac{l}{\mu r^2}\frac{p}{(\epsilon\cos\phi - 1)^2}\epsilon\sin\phi = \frac{l}{\mu p}\epsilon\sin\phi; \tag{8}$$

$$v_\infty = \frac{l}{\mu b} = \frac{l}{\mu p}\epsilon\sin\phi_\infty. \tag{9}$$

Thus we have $p = b\epsilon\sin\phi_\infty = b\epsilon\cos(\theta/2)$. We may find that

$$\epsilon^2 = \frac{p^2}{b^2\cos^2(\theta/2)} = \frac{1}{\sin^2(\theta/2)}, \qquad b = p\tan(\theta/2). \tag{10}$$

Then the differential cross section is

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \frac{b}{\sin\theta}\left|\frac{\mathrm{d}b}{\mathrm{d}\theta}\right| \\ &= \frac{b}{\sin\theta}\frac{p}{2\cos^2(\theta/2)} \\ &= \frac{b^2}{4}\frac{1}{\sin^2(\theta/2)\cos^2(\theta/2)}.\end{aligned} \tag{11}$$

The total energy of the orbit is

$$E = \frac{l^2}{2\mu p^2}(\epsilon^2 - 1) = \frac{k^2}{2l^2/\mu}(\epsilon^2 - 1) = \frac{k^2}{4Eb^2}\cot^2\left(\frac{\theta}{2}\right). \tag{12}$$

We have used the fact that $E = \mu v_\infty^2/2 = l^2/(2\mu b^2)$. We may also find an expression for b^2, then we have

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \frac{b^2}{4}\frac{1}{\sin^2(\theta/2)\cos^2(\theta/2)} \\ &= \frac{1}{4}\left(\frac{k}{2E}\right)^2\frac{1}{\sin^4(\theta/2)} \\ &= \left(\frac{Z_{Au}Z_\alpha e^2}{4E}\right)^2\frac{1}{\sin^4(\theta/2)}.\end{aligned} \tag{13}$$

- **(3 pt)** : Correct relation for $b(\theta)$.
- **(1 pt)** : Correct differential cross section definition.
- **(1 pt)** : Correct final expression.

Total sub-points : 5

(b) Sketch / plot the differential cross section as a function of θ.

[Solution]

See the figure.

- **(1 pt)** : Correct sketch.

Total sub-points : 1

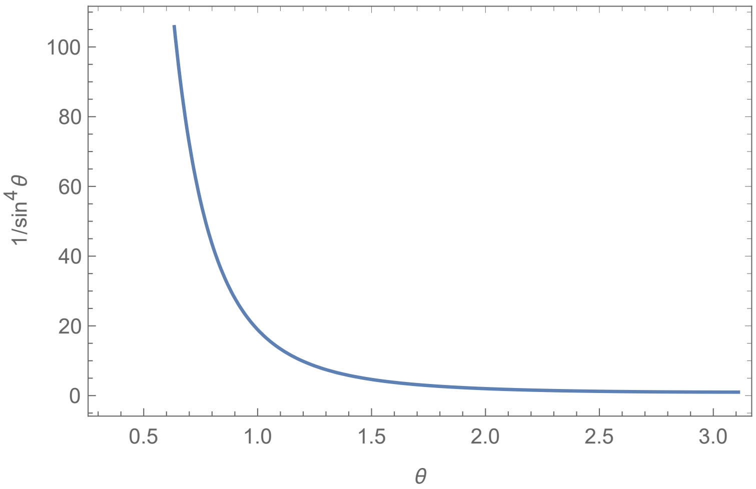


Figure 1: Differential cross section (scaled) versus the scattering angle.

(c) How close (minimum distance of approach) will an alpha particle with kinetic energy E come to the gold nucleus, in terms of all of the parameters of the problem?

[Solution]

As shown in the previous part, we found the trajectory of the α particle: $r = p/(\epsilon\cos\phi - 1)$. Thus the minimum distance to the gold nucleus is

$$\begin{aligned} r_{\min} =& \frac{p}{\epsilon - 1} \\ =& \frac{b\cot(\theta/2)}{1/\sin(\theta/2) - 1} \\ =& \frac{k\cot(\theta/2)}{2E}\frac{\cot(\theta/2)}{1/\sin(\theta/2) - 1} \\ =& \frac{k}{2E}\frac{1+\sin(\theta/2)}{\sin(\theta/2)}. \end{aligned} \tag{14}$$

We have used some conclusions in part (a). Apparently, when $\theta = \pi$, we have the minimum distance of

$$r_{\min} = \frac{k}{E} = \frac{Z_{Au}Z_\alpha e^2}{E}. \tag{15}$$

- **(3 pt)** : Correct result (simply comparing the kinetic and potential energy can also generate the answer). If used the method, 2 pt for expression of $r_{\min}$, 1 pt for correct answer.

Total sub-points : 3

(d) Numerically, how close (minimum distance of approach) will an alpha particle with 5.3 MeV kinetic energy come to the gold nucleus? You may use the following numerical values: $Z_{Au} = 79$ and $Z_\alpha = 2$, the classical electron radius $r_c = 2.817 \times 10^{-13}$ cm (a convenient way to express e^2), $m_e c^2 = 0.511$ MeV, $m_\alpha c^2 = 3728$ MeV, $m_{Au}c^2 = 183,471$ MeV.

[Solution]

With the given parameters we may find the $r_{\min} \doteq 4.3 \times 10^{-14}$m.

- **(1 pt)** : Correct result.

Total sub-points : 1

(e) Would the scattering cross section be different if the potential were attractive instead of repulsive?

[Solution]

No, since in calculations of part (a) we did not assumed k to be positive.

- **(1 pt)** : Correct statement.

Total sub-points : 1

(f) Qualitatively, what would you expect will happen to the angular distribution of scattering if the beam particles are sufficiently energetic to penetrate inside the nucleus? What minimum energy would you need to accomplish this, if The radius of a gold nucleus is approximately 7.5 fm = 7.5×10^{-13} cm.

[Solution]

There will be a drop in the $\mathrm{d}\sigma/\mathrm{d}\Omega$ - θ relation for large θ.

Apparently we have the minimum energy to be

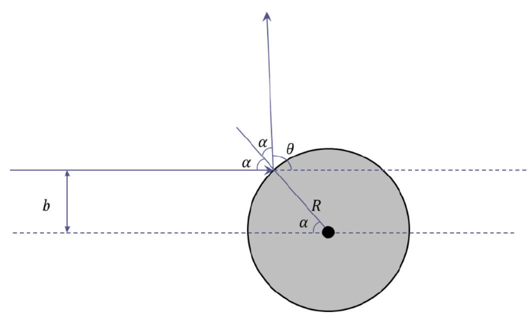
$$E_{\min} = \frac{k}{r_{\mathrm{Au}}} \doteq 30\mathrm{MeV}. \tag{16}$$

- **(1 pt)** : Correct statement.
- **(1 pt)** : Correct result.

Total sub-points : 2

2. **Cross section for hard sphere scattering**

A point projectile scatters off of a target rigid sphere of radius R, as depicted below from the side.



(a) Compute the differential scattering cross section. Integrate your result over all scattering angles to find the total cross section.

[Solution]

As shown in the figure, we have the apparent relations from the reflective law and the geometry:

$$b = R\sin\alpha = R\sin\left(\frac{\pi-\theta}{2}\right) = R\cos\left(\frac{\theta}{2}\right). \tag{17}$$

With the relation between the scattering angle θ and the scattering parameter b, we may compute the differential scattering cross section.

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \left|\frac{2\pi b\mathrm{d}b}{2\pi\sin\theta\mathrm{d}\theta}\right| \\ &= \frac{b}{\sin\theta}\cdot\left|\frac{\mathrm{d}b}{\mathrm{d}\theta}\right| \\ &= \frac{R\cos(\theta/2)}{\sin\theta}\cdot\frac{R}{2}\sin\left(\frac{\theta}{2}\right) \\ &= \frac{R^2}{4}.\end{aligned} \tag{18}$$

The differential cross section does not depend on θ. Finally, we may have the total cross section:

$$\begin{aligned}\sigma_{\text{tot}} &= \int \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\mathrm{d}\Omega \\ &= \int_0^\pi \frac{R^2}{4} \cdot 2\pi \sin\theta \mathrm{d}\theta \\ &= \pi R^2.\end{aligned} \tag{19}$$

- **(1 pt)** : Relation between θ and α.
- **(1 pt)** : Relation between b and θ.
- **(2 pt)** : Correct differential cross section: 1 pt for definition and 1 pt for calculation.
- **(2 pt)** : Correct total cross section: 1 pt for definition and 1 pt for calculation.

Total sub-points : 6

(b) Instead, assume that the projectile has a radius of R_b and the target has a radius of R_t. Again, compute the differential and total cross-sections.

[Solution]

Under this condition we have to adapt our geometrical relations. While the collision still follows the reflective law, the "effective" radius of the sphere has changed from R to $R' = R_t + R_b$, similarly we have

$$b = R' \sin\alpha = R' \sin\left(\frac{\pi - \theta}{2}\right) = R' \cos\left(\frac{\theta}{2}\right). \tag{20}$$

Since the new R' is a constant, we may simply replace R by R' to get the differential cross section.

$$\begin{aligned}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)' &= \frac{R'^2}{4} \\ &= \frac{1}{4}\left(R_t + R_b\right)^2.\end{aligned} \tag{21}$$

Finally, we may have the total cross section again:

$$\begin{aligned}\sigma'_{\text{tot}} &= \pi R'^2 \\ &= \pi \left(R_t + R_b\right)^2.\end{aligned} \tag{22}$$

- **(1 pt)** : Updated R to R' (or equivalently other methods).
- **(1 pt)** : Correct differential cross section.
- **(1 pt)** : Correct total cross section.

Total sub-points : 3

3. **Lab (fixed target) frame and center-of-mass frame**:

In the class lecture notes, review the computations that lead to relations between the scattering angle of the beam particle (of mass m) in the lab frame (where the heavier target particle, of mass M, starts out at rest) and in the center of mass frame:

$$\tan\theta_{\text{lab}} = \frac{\sin\theta_{\text{cm}}}{\cos\theta_{\text{cm}} + m/M},$$

and of the resulting cross section:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{lab}} = \frac{\sin\theta_{\text{cm}}}{\sin\theta_{\text{lab}}} \frac{d\theta_{\text{cm}}}{d\theta_{\text{lab}}} \left(\frac{d\sigma}{d\Omega}\right)_{\text{cm}}.$$

(a) Show that for equal mass particles, $\theta_{\text{lab}} = \theta_{\text{cm}}/2$.

[Solution]

When $m = M$, we have

$$\tan\theta_{\text{lab}} = \frac{\sin\theta_{\text{cm}}}{\cos\theta_{\text{cm}} + 1} = \frac{2\sin(\theta_{\text{cm}}/2)\cos(\theta_{\text{cm}}/2)}{2\cos^2(\theta_{\text{cm}}/2)} = \tan\left(\frac{\theta_{\text{cm}}}{2}\right). \quad (23)$$

So we conclude that $\theta_{\text{lab}} = \theta_{\text{cm}}/2$.

- **(1 pt)** : Correct proof of the relation.

Total sub-points : 1

(b) Show that for equal mass particles,

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{lab}} = 4\cos\theta_{\text{lab}}\left(\frac{d\sigma}{d\Omega}\right)_{\text{cm}}.$$

[Solution]

With $\theta_{\text{lab}} = \theta_{\text{cm}}/2$ we may have

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{lab}} = \frac{2\pi b}{2\pi\sin\theta_{\text{lab}}}\left|\frac{\mathrm{d}b}{\mathrm{d}\theta_{\text{lab}}}\right| = \frac{2\pi b}{2\pi\sin\theta_{\text{lab}}}\left|\frac{\mathrm{d}b}{\mathrm{d}\theta_{\text{cm}}/2}\right| = \frac{2\sin\theta_{\text{cm}}}{\sin\theta_{\text{lab}}}\frac{2\pi b}{2\pi\sin\theta_{\text{cm}}}\left|\frac{\mathrm{d}b}{\mathrm{d}\theta_{\text{cm}}}\right| = 4\cos\theta_{\text{lab}}\left(\frac{d\sigma}{d\Omega}\right)_{\text{cm}}. \quad (24)$$

- **(1 pt)** : Relation of the differential cross section in two frames.
- **(1 pt)** : Correct proof of the relation.

Total sub-points : 2

(c) For two hard spheres (as in the previous problem) of equal mass, verify that the total cross section in the lab frame is $\pi(R_b + R_t)^2$.

[Solution] In the center-of-mass frame, the two spheres collide with each other with equal velocity (since their masses are the same). Then they fly away in opposite directions by conserving the momentum and following the reflective law. The whole process is identical to the situation in problem 2 (b). Geometrically we have $b = (R_t + R_b)\cos(\theta/2)$ and also

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{cm}} = \frac{(R_b + R_t)^2}{4}. \quad (25)$$

We then find the differential cross section in the lab frame:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{lab}} = 4\cos\theta_{\text{lab}}\left(\frac{d\sigma}{d\Omega}\right)_{\text{cm}} = (R_b + R_t)^2\cos\theta_{\text{lab}}. \quad (26)$$

The total cross section in the lab frame is then

$$\begin{aligned}\sigma_{\text{tot}} =& \int \left(\frac{d\sigma}{d\Omega}\right)_{\text{lab}} \mathrm{d}\Omega_{\text{lab}} \\ =& \int_0^{\pi/2} (R_b + R_t)^2 \cos\theta_{\text{lab}} \cdot 2\pi \sin\theta_{\text{lab}} \mathrm{d}\theta_{\text{lab}} \\ =& \pi (R_b + R_t)^2.\end{aligned} \tag{27}$$

Note that since $\theta_{\text{lab}} = \theta_{\text{cm}}/2$ we have the integral range to be $(0, \pi/2)$.

- **(1 pt)** : Correct differential cross section in CM frame.
- **(1 pt)** : Correct differential cross section in LAB frame.
- **(1 pt)** : Correct integral and result.

Total sub-points : 3

4. **Hamiltonian treatment of nonlinear oscillator**: The Hamiltonian of a 1 degree of freedom system is

$$H = \tfrac{1}{2}p^2 - \tfrac{1}{2}q^2 + \tfrac{1}{4}q^4.$$

(a) Find the equations of motion $\dot{q}, \dot{p}$

[Solution]

q, p satisfy

$$\dot{q} = \frac{\partial H}{\partial p} = p, \tag{28}$$

$$\dot{p} = -\frac{\partial H}{\partial q} = q - q^3. \tag{29}$$

- **(1 pt)** : Equation for q.
- **(1 pt)** : Equation for p.

Total sub-points : 2

(b) Find the equilibria, and calculate the frequency of small oscillations about any stable equilibria.

[Solution]

The equilibria are given by $\dot{q} = \dot{p} = 0$ and are at $p = 0$ and $q = 0, \pm 1$. The equilibrium at $q = 0$ is unstable. The frequency about either of the other two is given by expansion, e.g. $q = 1 + \delta q, p = \delta p$ and then to linear order

$$\dot{\delta q} = \delta p \tag{30}$$

$$\dot{\delta p} = -2\delta q. \tag{31}$$

This gives simple harmonic motion at frequency $\sqrt{2}$.

- **(1 pt)** : Find three equilibria.
- **(1 pt)** : Identify the stable and unstable equilibria.
- **(1 pt)** : Correct perturbation differential equation for δp and δq.
- **(1 pt)** : Find the frequency.

Total sub-points : 4

(c) Sketch a variety of trajectories of the dynamics in phase space (q, p) (including the direction they are traversed) and indicate any equilibria. Make sure you include enough trajectories to illustrate: small amplitude oscillations about stable equilibria; the behavior near any unstable equilibria; large amplitude oscillations; and the homoclinic or heteroclinic orbit(s) (i.e. the infinite period orbit(s) starting and returning to an unstable equilibrium).

[Solution] See the figure. To make the plot, one may solve for the equation of motion and then plot with parametric t. Alternatively, since the Hamiltonian here does not explicitly depend on time, it is conserved for a single trajectory. One may plot the three conditions as shown in the figure with contours and then add arrows based on the equation of motion.

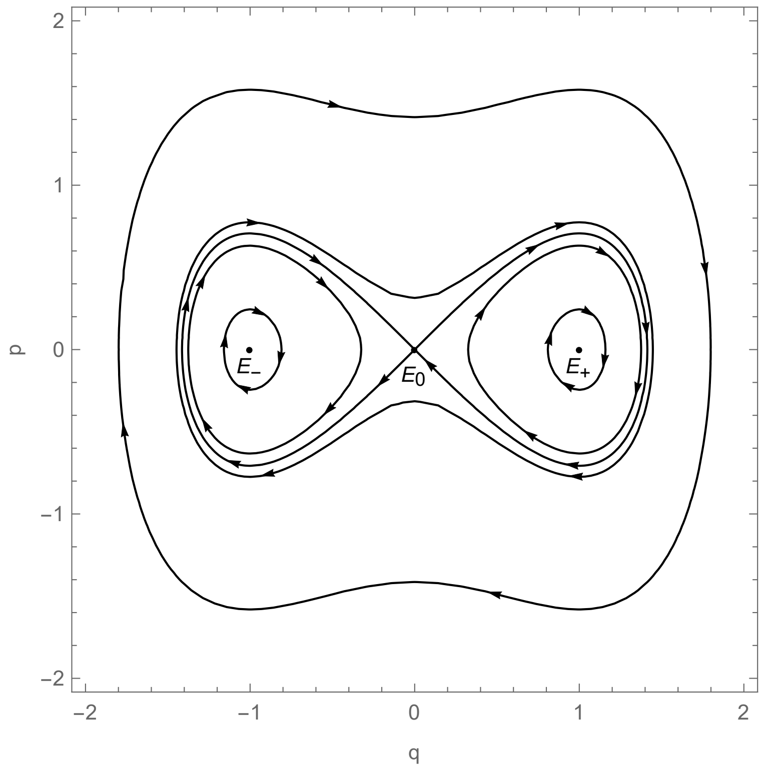


Figure 2: Phase diagram of the Hamiltonian equation of motion. Here $E_\pm$ are two stable equilibria, while the E_0 is unstable. Near $E_\pm$, we see small amplitude oscillations about stable equilibria; across E_0 is the homoclinic orbit; the most outer contour is an example of the large amplitude oscillation. The figure also shows the orbits closely near E_0.

- **(3 pt)** : Include and identify three conditions of the trajectories.
- **(1 pt)** : Label the stable and unstable equilibria.
- **(2 pt)** : Correct arrows or equivalent descriptions.

Total sub-points : 6