Physics 106a: Assignment 5 *(Solutions prepared by Alvin Li)*

8 November, 2019; due 5pm Friday, 15 November in the “Ph106 In Box” in East Bridge mailbox.

One Dimensional Systems: Central Forces and the Kepler Problem

Reading: Hand and Finch Chapter 4.

Problems

1. **Binary-sun solar system**: Consider a binary pair of identical suns of mass $M$ orbiting in the $x-y$ plane in an orbit centered at the origin. The gravitational constant is $G$. Now add a planet of small mass $m$ with an initial condition on the $z$ axis above the center of mass of the two suns and with a velocity in the $z$ direction; by the symmetry of the system the small mass will remain on the $z$ axis and suns will have equal $z$ coordinates and their center of mass will also remain on the $z$ axis. We can choose as coordinates describing the dynamics: the $z$ coordinate of the planet $z$; the $z$ coordinate of the two suns $Z$; and the polar coordinates $(R,\theta)$ giving the $x,y$ coordinates of the suns $(\pm R\cos \theta, \pm R\sin \theta)$ — see figure.

(a) What is the Lagrangian of the system in terms of these coordinates and their time derivatives?

**[Solution]**

For each of the large suns $i$, the kinetic energy is

$$T_{M_i} = \frac{1}{2} M \dot{R}_i^2 + \frac{1}{2} M Z_i^2 + \frac{1}{2} M R^2 \dot{\theta}_i^2,$$

and for the small planet, the kinetic energy is

$$T_m = \frac{1}{2} m \dot{z}^2.$$ (2)

The potential energy comes from three contributions: The gravitational potential between $M_1$ and $M_2$ (separated by a distance of $2R$), that between $M_1$ and $m$ (separated by a distance of $\sqrt{R^2 + (z-Z)^2}$), and lastly that between $M_2$ and $m$ (separated by a distance of $\sqrt{R^2 + (z-Z)^2}$). Mathematically, we have

$$V = \underbrace{-GM_1^2}_{\text{GPE between } M_1 \text{ and } M_2} + \underbrace{-GM_2 m}_{\text{GPE between } M_2 \text{ and } m} - 2 \times \underbrace{\frac{GMm}{\sqrt{R^2 + (z-Z)^2}}}_{\text{GPE between } M_1 \text{ and } m}$$ (3)

Hence, the Lagrangian is given by

$$\mathcal{L} = 2 \times T_{M_i} + T_m - V$$

$$= M \dot{R}^2 + M \dot{Z}^2 + M R^2 \dot{\theta}^2 + \frac{1}{2} m \dot{z}^2 + \frac{GM}{2R} + \frac{2GMm}{\sqrt{R^2 + (z-Z)^2}}$$ (5)
and hence we can write the Lagrangian as
\[
L = M\dot{R}^2 + M\dot{Z}^2 + MR^2\dot{\theta}^2 + \frac{1}{2}m\dot{z}^2 + \frac{GM^2}{2R} + \frac{2GMm}{\sqrt{R^2 + (z - Z)^2}}
\]
Together with the new coordinate
\[
z = \frac{m}{m + 2M}z + \frac{2M}{m + 2M}Z.
\]
which coincide with the planet. The overall center of mass of the system is given by
\[
Z_{\text{com}} = Q = \frac{m}{m + 2M}z + \frac{2M}{m + 2M}Z.
\]

We notice that the Lagrangian:
\[
\mathcal{L} = M\dot{R}^2 + M\dot{Z}^2 + MR^2\dot{\theta}^2 + \frac{1}{2}m\dot{z}^2 + \frac{GM^2}{2R} + \frac{2GMm}{\sqrt{R^2 + (z - Z)^2}}
\]
does not depend on explicitly on the coordinate \(\theta\). This means that \(\theta\) is an ignorable / cyclic coordinate, and hence the conjugate (generalized) momentum corresponding to \(\theta\), which, in this case, is the total angular momentum \(l\) of the system, is a conserved quantity. We have
\[
l = \frac{\partial L}{\partial \dot{\theta}} = 2MR^2\dot{\theta} = \text{constant}
\]
The center of mass of the two suns lies on the \(z\) axis (and is of the same height as the two suns), which coincide with the planet. The overall center of mass of the system is given by
\[
Z_{\text{com}} = Q = \frac{m}{m + 2M}z + \frac{2M}{m + 2M}Z.
\]
Together with the new coordinate
\[
q = z - Z,
\]
we can solve for \(z\) and \(Z\) in terms of \(q\) and \(Q\) as
\[
\begin{cases}
z = Q + \frac{2M}{m + 2M}q \\ Z = Q - \frac{m}{m + 2M}q
\end{cases}
\]
and hence we can write the Lagrangian as
\[
\mathcal{L} = M\dot{R}^2 + M\dot{Z}^2 + MR^2\dot{\theta}^2 + \frac{1}{2}m\dot{z}^2 + \frac{GM^2}{2R} + \frac{2GMm}{\sqrt{R^2 + q^2}}
\]
\[
= M\dot{R}^2 + M\left(\dot{Q} - \frac{m}{m + 2M}\dot{q}\right)^2 + MR^2\dot{\theta}^2 + \frac{1}{2}m\left(\dot{Q} + \frac{2M}{m + 2M}\dot{q}\right)^2 + \frac{GM^2}{2R} + \frac{2GMm}{\sqrt{R^2 + q^2}}
\]
\[
= M\dot{R}^2 + \left(M + \frac{m}{2}\right)\dot{Q}^2 + \left[M\left(\frac{m}{m + 2M}\right)^2 + \frac{m}{2}\left(\frac{2M}{m + 2M}\right)^2\right]\dot{q}^2 + MR^2\dot{\theta}^2 + \frac{GM^2}{2R} + \frac{2GMm}{\sqrt{R^2 + q^2}}
\]
\[
= M\dot{R}^2 + \frac{1}{2}\left(m + 2M\right)\dot{q}^2 + \frac{Mm(m + 2M)}{(m + 2M)^2}\dot{q}^2 + MR^2\dot{\theta}^2 + \frac{GM^2}{2R} + \frac{2GMm}{\sqrt{R^2 + q^2}}
\]
\[
= M\dot{R}^2 + \frac{1}{2}\left(m + 2M\right)\dot{q}^2 + \frac{2Mm}{m + 2M}\dot{q}^2 + MR^2\dot{\theta}^2 + \frac{GM^2}{2R} + \frac{2GMm}{\sqrt{R^2 + q^2}}
\]
\( \mathcal{L} = M\dot{R}^2 + \frac{1}{2} (m+2M)\dot{Q}^2 + \frac{1}{2} \left( \frac{1}{2M} + \frac{1}{m} \right) q^2 + MR^2 \dot{\theta}^2 + \frac{GM^2}{2R} + \frac{2GMm}{\sqrt{R^2 + q^2}} \) \hspace{1cm} (17)

\[ = M\dot{R}^2 + \frac{1}{2} (m+2M)\dot{Q}^2 + \frac{1}{2} \mu q^2 + MR^2 \dot{\theta}^2 + \frac{GM^2}{2R} + \frac{2GMm}{\sqrt{R^2 + q^2}} \] \hspace{1cm} (18)

where \( \mu \) is the reduced mass of the system defined by

\[ \frac{1}{\mu} = \frac{1}{2M} + \frac{1}{m} \] \hspace{1cm} (19)

Now, since the Lagrangian does not depend explicitly on the coordinate \( Q \), \( Q \) is therefore a cyclic / ignorable coordinate, and hence the conjugate momentum, which is the linear momentum \( P_Q \) along the \( z \) direction, is a conserved quantity. Mathematically, we have

\[ P_Q = \frac{\partial \mathcal{L}}{\partial \dot{Q}} = (m+2M)\dot{Q} = \text{constant}, \] \hspace{1cm} (20)

Using Lagrange equation

\[ \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0, \] \hspace{1cm} (21)

we have the equation of motion for the \( R \) component as

\[ \frac{d}{dt} (2M\ddot{R}) - \left( 2M\dot{\theta}^2 - \frac{GM^2}{2R^2} - \frac{2GMmR}{(R^2 + q^2)^{3/2}} \right) = 0 \] \hspace{1cm} (22)

\[ 2M\ddot{R} - 2MR\dot{\theta}^2 + \frac{GM^2}{2R^2} + \frac{2GMmR}{(R^2 + q^2)^{3/2}} = 0 \] \hspace{1cm} (23)

\[ 2M\ddot{R} - \frac{l^2}{2MR^3} + \frac{GM^2}{2R^2} + \frac{2GMmR}{(R^2 + q^2)^{3/2}} = 0 \] \hspace{1cm} (24)

and the equation of motion for the \( q \) component as

\[ \frac{d}{dt} (\mu \ddot{q}) - \left( -\frac{2GMmq}{(R^2 + q^2)^{3/2}} \right) = 0 \] \hspace{1cm} (25)

\[ \mu \dddot{q} + \frac{2GMmq}{(R^2 + q^2)^{3/2}} = 0 \] \hspace{1cm} (26)

**Supplementary box 5.1: Conserved quantities for which coordinate?**

Some readers may try to rewrite the Lagrangian as

\[ \mathcal{L} = M\dot{R}^2 + \frac{P_Q^2}{2(m+2M)} + \frac{1}{2} \mu q^2 + \frac{l^2}{4MR^2} + \frac{GM^2}{2R} + \frac{2GMm}{\sqrt{R^2 + q^2}} \] \hspace{1cm} (27)

If they do so, when they apply the Euler-Lagrange equation they will find that the sign of the term \( \frac{l^2}{4MR^2} \) will be incorrect. What is wrong here?

Readers should note that when we are going through the logic that “since the Lagrangian is not depending explicitly on a specific coordinate \( q_i \), and hence the conjugate momentum \( p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \) corresponding to that coordinate is a **conserved quantity**”, what we really mean is that the conjugate momentum is a conserved quantity in time (i.e. it is not changing with time). However, it does not immediately mean that the conjugate momentum is not changing with respect to the *other coordinates*! For example, consider the total angular momentum

\[ l = 2MR^2 \dot{\theta}. \] \hspace{1cm} (28)
It is obvious that the angular momentum is changing with respect to the coordinate \( R \). In particular,

\[
\frac{\partial l}{\partial R} = 4MR\dot{\theta} = \frac{2l}{R} \neq 0 \tag{29}
\]

And hence this explains why one will not get the correct answer if they substitute \( l \) into the Lagrangian before applying the Lagrange equation. Again, readers are reminded to be careful about the “conserved quantity” we are talking about here are only conserved in time.

- (1 pt) : Stating that \( \mathcal{L} \) does not depend on \( \theta \) explicitly, and hence \( \theta \) is ignorable / cyclic, and its conjugate momentum (the total angular momentum) is a conserved quantity.
- (1 pt) : Solving \( l \) in terms of \( M \), \( R \) and \( \theta \).
- (2 pt) : Introducing the coordinate \( Q \), and solve \( z \) and \( Z \) in terms of \( q \) and \( Q \).
- (1 pt) : Stating that \( \mathcal{L} \) does not depend on \( Q \) explicitly, and hence \( Q \) is ignorable / cyclic, and its conjugate momentum (the linear momentum along the \( z \) direction) is a conserved quantity.
- (1 pt) : Finding and expressing the equation of motion for \( r \) in terms of \( l \) and the other parameters. Note that if the term involving \( l \) does not have a correct sign, credit SHOULD NOT be awarded.
- (1 pt) : Finding and expressing the equation of motion for \( q \) in terms of the other parameters.

Total sub-points : 8

(c) In the limit of small planetary mass \( m \ll M \) we can ignore the effect of the planet on the motion of the suns. Find the explicit solution for the motion of the suns \( R(t) \) for orbits with small eccentricity \( \epsilon \). Write your solution as circular motion plus a term proportional to \( \epsilon \).

[Solution]

If \( \frac{m}{M} \rightarrow 0 \), the equation of motion for \( R \) can be written as

\[
2M\ddot{R} - \frac{l^2}{2M^3R_0^3} + \frac{GM^2}{2R^2} + \frac{2GMmR}{(R^2 + q^2)^{\frac{3}{2}}} = 0 \tag{30}
\]

\[
\frac{2}{M} \ddot{R} - \frac{l^2}{2M^3R_0^3} + \frac{G}{2R^2} + \frac{2GR}{(R^2 + q^2)^{\frac{3}{2}}} \approx 0 \tag{31}
\]

\[
\frac{2}{M} \ddot{R} - \frac{l^2}{2M^3R_0^3} + \frac{G}{2R^2} = 0. \tag{32}
\]

For circular orbit, we have \( R = R_0 = \text{constant} \) and \( \dot{R} = \ddot{R} = 0 \), hence the equation of motion for \( R \) can be written as

\[
\frac{l^2}{2M^3R_0^3} + \frac{G}{2R_0^2} = 0 \tag{33}
\]

\[
R_0 = \frac{l^2}{GM^3} \tag{34}
\]

For a slightly non-circular orbit, we can write

\[
R(t) = R_0(1 + \epsilon f(t)) \tag{35}
\]

where \( \epsilon \), the eccentricity, is a small quantity, and \( f(t) \) embeds the information of the time evolution of the orbit. We call \( \epsilon f(t) \) as the perturbation term to the orbit.
Note that under the assumption that $\epsilon$ is small, we have
\[
\frac{1}{R^3} = \frac{1}{R_0^3} (1 + \epsilon g)^{-3} = \frac{1}{R_0^3} (1 - 3\epsilon f + O(\epsilon^2)) \approx \frac{1}{R_0^3} (1 - 3\epsilon f)
\] (36)

and
\[
\frac{1}{R^2} = \frac{1}{R_0^2} (1 + \epsilon g)^{-2} = \frac{1}{R_0^2} (1 - 2\epsilon f + O(\epsilon^2)) \approx \frac{1}{R_0^2} (1 - 2\epsilon f).
\] (37)

Putting $R(t)$ back to the equation of motion for $R$ (equation 34), we have
\[
\frac{2R_0\epsilon}{M} \ddot{f} - \frac{l^2}{2M^3R_0^4} (1 - 3\epsilon f) + \frac{G}{2R_0^2} (1 - 2\epsilon f) = 0
\] (38)
\[
\frac{2R_0\epsilon}{M} \ddot{R} - \frac{G}{2R_0^2} (1 - 3\epsilon f) + \frac{G}{2R_0^2} (1 - 2\epsilon f) = 0
\] (39)
\[
\frac{2R_0\epsilon}{M} \ddot{f} - \frac{G\epsilon f}{2R_0^2} = 0
\] (40)
\[
\ddot{f} = -\frac{GM}{4R_0^2} f
\] (41)
\[
f = A \cos \left( \sqrt{\frac{GM}{2R_0}} t + \phi \right)
\] (42)

where $A$ and $\phi$ are constants to be determined by initial conditions.

The general solution is given by
\[
R(t) = R_0 \left( 1 + A\epsilon \cos \left( \sqrt{\frac{GM}{2R_0}} t + \phi \right) \right)
\] (43)

Assuming the initial radial velocity of the orbit of the suns is $\dot{R} = 0$, we have $\dot{\phi} = 0$. Hence the solution for $R$ can be written as
\[
R(t) = R_0 \left( 1 + A\epsilon \cos \left( \sqrt{\frac{GM}{2R_0}} t + \phi \right) \right)
\] (44)

If we further assume that the initial radius of the orbit is $R = R_0 (1 + \epsilon)$, then we have $A = 1$ and hence the solution can be written as
\[
R(t) = R_0 \left( 1 + \epsilon \cos \left( \sqrt{\frac{GM}{2R_0\sqrt{R_0}}} t \right) \right)
\] (45)

- (1 pt): Showing / Arguing that the fourth term in the equation of motion for $R$ is negligible if $m << M$.
- (2 pt): Solving $R_0$ in terms of $l$, $G$ and $M$ for the circular orbit case.
- (1 pt): Expanding $R(t)$ as the circular orbital radius $R_0$ plus a perturbation term $\epsilon f$.
- (2 pt): Expanding $\frac{1}{R^2}$ and $\frac{1}{R^3}$ up to the first order term of $\epsilon$.
- (3 pt): Putting $R(t)$ back into the equation of motion, solving the equation, and showing that $f$ is a sinusoidal function.
- (1 pt): Correct final expression for $R(t)$. Note that students do not have to assume any initial conditions, but should keep the undetermined constants in the solution.

Total sub-points: 10
(d) Hence show for these limits, keeping terms linear in \( \epsilon \), that the equation of motion for \( q \) is a nonlinear oscillator (oscillator with a non-quadratic potential) driven by a term \( \epsilon f(q) \cos \Omega t \) with 2\( \pi / \Omega \) the period of the solar motion, and find the form of \( f(q) \).

**Solution**

In the regime of \( m \ll M \), we have

\[
\frac{1}{\mu} = \frac{1}{m} + \frac{1}{2M} \approx \frac{1}{m}
\]

Hence we have the equation of motion for \( q \) as

\[
m\ddot{q} + \frac{2GMm}{(R^2 + q^2)^{\frac{3}{2}}} = 0
\]

Note that

\[
(R^2 + q^2)^{-\frac{3}{2}} = \left( R_0^2 \left( 1 + A\epsilon \cos \left( \frac{\sqrt{GM}}{2R_0\sqrt{R_0}} t + \phi \right) \right)^2 + q^2 \right)^{-\frac{3}{2}}
\]

\[
\approx (R_0^2 + q)^{-\frac{3}{2}} \left( 1 + \frac{2R_0^2}{R_0^2 + q^2} A\epsilon \cos \left( \frac{\sqrt{GM}}{2R_0\sqrt{R_0}} t + \phi \right) \right)^{-\frac{3}{2}}
\]

\[
= (R_0^2 + q)^{-\frac{3}{2}} \left( 1 - \frac{3}{2} \frac{2R_0^2}{R_0^2 + q^2} A\epsilon \cos \left( \frac{\sqrt{GM}}{2R_0\sqrt{R_0}} t + \phi \right) + O(\epsilon^2) \right)
\]

\[
\approx \frac{1}{(R_0^2 + q^2)^{\frac{3}{2}}} - \frac{3R_0^2}{(R_0^2 + q^2)^{\frac{3}{2}}} A\epsilon \cos \left( \frac{\sqrt{GM}}{2R_0\sqrt{R_0}} t + \phi \right).
\]

Hence, the equation of motion for \( q \) can be written as

\[
m\ddot{q} + \frac{2GMm}{(R_0^2 + q^2)^{\frac{3}{2}}} q - \frac{6GMmR_0^3}{(R_0^2 + q^2)^{\frac{3}{2}}} A\epsilon \cos \left( \frac{\sqrt{GM}}{2R_0\sqrt{R_0}} t + \phi \right) = 0
\]

Using the initial conditions, we have \( A = 1 \) and \( \phi = 0 \), which further simplifies the equation of motion to the form

\[
m\ddot{q} + \frac{2GMm}{(R_0^2 + q^2)^{\frac{3}{2}}} q - \frac{6GMmR_0^3}{(R_0^2 + q^2)^{\frac{3}{2}}} \epsilon q \cos \left( \frac{\sqrt{GM}}{2R_0\sqrt{R_0}} t \right) = 0
\]

We can see that the equation of motion is composed of a usual “EOM for simple harmonic oscillator” and a non-linear “driving force” in the form \( \epsilon f(q) \cos \Omega t \) \( (\Omega = \frac{\sqrt{GM}}{2R_0\sqrt{R_0}}) \), where \( f(q) \) is given by

\[
\frac{6GMmR_0^3}{(R_0^2 + q^2)^{\frac{3}{2}}} \epsilon q.
\]

If you want to learn more about the complex dynamics (chaos) of this system, you can Google the “Sloshnikov Problem”. One good reference is Hevia and Ranada, Eur. J. Phys. **17**, 295 (1996).

- **(1 pt)**: Arguing that for \( m \ll M, \mu \approx m \).
- **(2 pt)**: Expanding \((R^2 + q^2)^{-\frac{3}{2}}\) up to first order of \( \epsilon \).
- **(1 pt)**: Correct final expression for the equation of motion of \( q \).
- **(1 pt)**: Correct expression for \( f(q) \).

Total sub-points : 5
For your interest only: The dynamics of this system given by the equation of motion you have calculated turns out to be complex even in the sensible limit \( m \ll M \). One way of illustrating the complexity is to consider the crossing times \( \tau_1, \tau_2 \ldots \) for which the planet crosses the plane of the orbit of the suns. It can be shown that for any chosen sequence of numbers (e.g. random numbers) an initial condition can be found such that the \( \tau_i \) reproduce this sequence, with escape to infinity corresponding to the end of a finite sequence. Thus no matter how many \( \tau_i \) are measured from observation of such a system, no prediction can be made for the next \( \tau \) in the dynamics! These chaotic dynamics are a demonstration of the difficulty of analyzing the 3 body problem in gravitational physics.

2. The Yukawa potential: A particle of mass \( m \) moves in a force field described by the Yukawa potential \( V(r) = -\frac{k}{r} \exp(-\frac{r}{a}) \), where \( k \) and \( a \) are positive. In the limit \( a \to \infty \), this reduces to the Kepler problem. When \( a \) is finite, this results in an attenuation of the potential, and force, for \( r \gg a \); i.e., a short-range potential. It is commonly used to describe the potential between nucleons in the nucleus, with \( a \approx 10^{-15} \) m.

(a) Write the equations of motion and reduce them to the equivalent one-dimensional problem, in terms of an effective potential.

**Solution**

Assuming that the motion of the particle is restricted to a plane, we can use \((r, \theta)\) to describe the position of the particle. The Lagrangian for this problem is then given by

\[
\mathcal{L} = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 + \frac{k}{r} \exp \left( -\frac{r}{a} \right) = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 - V(r) \tag{56}
\]

Since the Lagrangian does not depend on \( \theta \) explicitly, \( \theta \) is ignorable / cyclic and hence its conjugate momentum (which is the angular momentum \( l \)) is conserved. We have

\[
l = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m r^2 \dot{\theta} \tag{57}
\]

Using the Lagrange equation

\[
\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0, \tag{58}
\]

the equation of motion for the \( r \) component is

\[
\frac{d}{dt} (m \ddot{r}) - \left( m r \dot{\theta}^2 - \frac{\partial V(r)}{\partial r} \right) = 0 \tag{59}
\]

\[
m \ddot{r} - m r \dot{\theta}^2 + \frac{\partial V(r)}{\partial r} = 0 \tag{60}
\]

\[
m \ddot{r} - \frac{l^2}{r^3} + \frac{\partial V(r)}{\partial r} = 0 \tag{61}
\]

\[
m \ddot{r} - \left( \frac{\partial}{\partial r} \left( -\frac{l^2}{2mr^2} - \frac{\partial V(r)}{\partial r} \right) \right) = 0 \tag{62}
\]

\[
m \ddot{r} - \frac{\partial}{\partial r} \left( -\frac{l^2}{2mr^2} - V(r) \right) = 0 \tag{63}
\]

\[
m \ddot{r} + \frac{\partial V_{\text{eff}}}{\partial r} = 0 \tag{64}
\]

where the effective potential is given by

\[
V_{\text{eff}} = \frac{l^2}{2mr^2} + V(r) = \frac{l^2}{2mr^2} - \frac{k}{r} \exp \left( -\frac{r}{a} \right) \tag{65}
\]

• (2 pt): Stating that the angular momentum \( l \) is a conserved quantity (either through the equation of motion, or by observing that \( \theta \) is a cyclic / ignorable coordinate), and finding the correct expression for \( l \).
• (1 pt): Using the Lagrange equation to find an equation of motion for \( r \) in terms of \( r \) and \( \theta \) (and \( V(r) \)).
• (1 pt): Replacing the term involving \( \dot{\theta} \) by \( l \).
• (2 pt): Writing the equation of motion for \( r \) in terms of the effective potential \( V_{\text{eff}} \), and writing down the correct expression for the effective potential.

Total sub-points: 6

(b) Use the effective potential to identify and discuss the qualitative nature of orbits for different values of the energy \( E \) and the angular momentum \( \ell \).

[Solution]

The effective potential is

\[
V_{\text{eff}} = \frac{l^2}{2mr^2} + V(r) = \frac{l^2}{2mr^2} - \frac{k}{r} \exp \left( -\frac{r}{a} \right) \tag{66}
\]

and the total energy is

\[
E = \frac{1}{2} \frac{m}{r^2} + \frac{l^2}{2mr^2} + V(r) = \frac{1}{2} \frac{m}{r^2} + \frac{l^2}{2mr^2} - \frac{k}{r} \exp \left( -\frac{r}{a} \right) \tag{67}
\]

As usual, we compare the total energy \( E \) with the effective potential \( V_{\text{eff}} \). The difference between \( E \) and \( V_{\text{eff}} \) (at different \( r \)) gives the radial kinetic energy, i.e. \( \frac{1}{2} m \dot{r}^2 \), of the particle (at that particular position). If there is a minimum point for \( V_{\text{eff}} \), there will be bound states for the particle. Otherwise, there will be no bound states!

We look for solutions such that \( V_{\text{eff}} \) has local extremum points:

\[
\frac{dV_{\text{eff}}}{dr} = -\frac{l^2}{mr^3} + \frac{k}{r^2} \exp \left( -\frac{r}{a} \right) + \frac{k}{ar} \exp \left( -\frac{r}{a} \right) = 0 \tag{68}
\]

\[
\frac{l^2k}{mr^3} \left[ \frac{1}{k} - \left( \frac{m}{l^2} \frac{m}{al^2} r^2 \right) \exp \left( -\frac{r}{a} \right) \right] = 0 \tag{69}
\]

\[
\frac{1}{k} - \left( \frac{m}{l^2} r^2 + \frac{m}{al^2} r^2 \right) \exp \left( -\frac{r}{a} \right) = 0 \tag{70}
\]

If we write \( \frac{r}{a} = x \) and \( b = \frac{l^2}{ma} \), we have

\[
\frac{1}{k} - \left( \frac{x}{b} + \frac{x^2}{b} \right) e^{-x} = 0 \tag{71}
\]

\[
x(x+1)e^{-x} = \frac{b}{k} = \frac{l^2}{mka} \tag{72}
\]

From the graph of \( f(x) = x(x+1)e^{-x} \) from \( x = 0 \),
we can see that if $\frac{I^2}{mka} \geq 0$ is small enough, we will have two solutions, $x_1 = \frac{r_1}{a}$ and $x_2 = \frac{r_2}{a}$ (where we take for convenience $x_1 < x_2$), for $\frac{\partial V_{eff}}{\partial r} = 0$. The slope at these points on the curve $f(x)$ denotes the nature of these local extremum points for $V_{eff}$. In particular, we see that $f'(x_1) > 0$, meaning that $x_1$ is a local minimum, and $f'(x_2) < 0$, meaning that $x_2$ is a local maximum point. In this case there can be bound states:

1. Firstly, we note that for $0 < E < V_{eff}(x_2)$, both bound states and unbound states can exist. For unbound states, they are either hyperbolic (if $E < V_{eff}(x_2)$) or parabolic (if $E = V_{eff}(x_2)$). See the figure below for a graphical explanation. For items (2) to (4), we will focus on the bound states, but readers are asked not to forget that unbound states can also happen.

2. If $E = V_{eff}(x_1) = V_{eff}^{\text{min}}$, the radial kinetic energy of the particle will be 0 at all times, and $r$ can only be a fixed value at $r = ax_1 = r_1$. This corresponds to a bound circular orbit.

3. If $V_{eff}(x_1) < E < V_{eff}(x_2)$, the particle will stay in a bound elliptical orbit with aphelion and perihelion distance equal to intersection points of $y = E$ and $y = V_{eff}$.

4. If $E = V_{eff}(x_2)$, there will be an unstable bound state (Still a circular orbit, but unstable) for the particle at $r = ax_2 = r_2$. A slight deviation of the particle from the position $r = r_2$ will lead to an unbound state, which corresponds to a hyperbolic orbit.

5. Finally, if $E > V_{eff}(x_2)$, we will have an unbound orbit, i.e. a hyperbolic orbit.

As $\frac{I^2}{mka}$ gets larger, the two solutions $x_1$ and $x_2$ will come closer and closer to each other until they finally merge into a single point with slope $f'(x_1 = x_2) = 0$. This corresponds to the point of inflexion for $V_{eff}$. At this point, there will only an unstable bound state, and at other energies $E$ we will have unbound states only.
When \( \frac{l^2}{m^2a} \) is larger than the maximum value of \( f(x) \) (for \( x \geq 0 \), there will be no solutions for \( \frac{\partial V_{\text{eff}}}{\partial r} = 0 \). This means that only unbound states can exist. In other words, there will be no bound states.

• (1 pt) : Correct expression for the total energy \( E \). This is important since we are comparing \( E \) and \( V_{\text{eff}} \).

• (1 pt) : Differentiating \( V_{\text{eff}} \) with respect to \( r \) to find the local extremum points of \( V_{\text{eff}} \). Comparing \( E \) and \( V_{\text{eff}} \).

• (2 pt) : Finding the relationship between the value of \( \frac{l^2}{m^2a} \) (if students do not use the parametrization suggested in the solution, credit should still be given with plausible comparisons between \( r \) and \( l \), \( k \), \( a \) and \( m \) in the equation for \( \frac{\partial V_{\text{eff}}}{\partial r} = 0 \) and the number of local extremum points for \( V_{\text{eff}} \).

• (4 pt) : For the two local extremum points case, explaining the nature of orbits for (1) \( E = V_{\text{eff}}(x_1) \), (2) \( V_{\text{eff}}(x_1) < E < V_{\text{eff}}(x_2) \), (3) \( E = V_{\text{eff}}(x_2) \) and (4) \( E > V_{\text{eff}}(x_2) \) AND \( 0 < E < V_{\text{eff}}(x_2) \). One point for each item. If the student does not state that there could be bound state even if \( E < V_{\text{eff}}(x_2) \), awards only 0.5 point for the last item.

• (2 pt) : Explaining that when the two local extremum points merge into one, the point becomes a point of inflexion and hence only one unstable bound state can exist, otherwise the orbit will be unstable.

• (1 pt) : Stating there will only be unbound states if there is no solution to \( \frac{\partial V_{\text{eff}}}{\partial r} = 0 \).

\[ \text{Total sub-points : 11} \]

(c) In the limit \( a \to \infty \) (the Kepler problem), give expressions for the radius of the circular orbit \( r_c \), the orbital frequency (inverse of the orbital period), and the energy, in terms of \( l \), \( k \), and \( m \).

[Solution]

As \( a \to \infty \), we have

\[ V \approx -\frac{k}{r} \] \hspace{1cm} (73)

and

\[ V_{\text{eff}} \approx \frac{l^2}{2mr^2} - \frac{k}{r} \] \hspace{1cm} (74)
The equation of motion for $r$ becomes

$$m \ddot{r} - \frac{l^2}{mr^3} + \frac{\partial}{\partial r} \left( - \frac{k}{r} \right) = 0 \quad (75)$$

$$m \ddot{r} - \frac{l^2}{mr^3} + \frac{k}{r^2} = 0 \quad (76)$$

For circular orbit, $r = r_c$ = Constant, and hence $\dot{r} = \ddot{r} = 0$. We have

$$- \frac{l^2}{mr_c^3} + \frac{k}{r_c^2} = 0 \quad (77)$$

$$r_c = \frac{l^2}{mk} \quad (78)$$

Recall that the angular momentum of the particle is

$$l = mr^2 \dot{\theta} = mr^2 \omega \quad (79)$$

where

$$\omega = \frac{l}{mr^2} = \frac{mk^2}{l^3} \quad (80)$$

is the orbital angular velocity. From this, the orbital frequency is given by

$$f = \frac{\omega}{2\pi} = \frac{mk^2}{2\pi l^3} \quad (81)$$

The energy is given by

$$E = \frac{1}{2} \dot{r}^2 + V_{\text{eff}}(r = r_c) = \frac{l^2}{2mr_c^2} - \frac{k}{r_c} = \frac{mk^2}{2l^2} - \frac{mk^2}{l^2} = - \frac{mk^2}{2l^2} \quad (82)$$

as expected.

**Supplementary Box 5.2 : Why “as expected”?**

The energy $E$ for the circular orbit we calculated is

$$E = - \frac{mk^2}{2l^2} = -T = \frac{1}{2} V \quad (83)$$

Let’s consider the situation when an object $m$ is orbiting around a planet of mass $M$ (with $M >> m$) in a circular orbit with radius $R$. We have the gravitational force acting on the object by the planet equals to the centripetal force, i.e.

$$\frac{mv^2}{R} = \frac{GMm}{R^2} \quad (84)$$

$$v = \sqrt{\frac{GM}{R}} \quad (85)$$

The kinetic energy of the particle is

$$T = \frac{1}{2} m|v|^2 = \frac{GMm}{2R} \quad (86)$$

The potential energy of the particle is

$$V = - \frac{GMm}{R} = -2T \quad (87)$$

Hence, we see that

$$T = -2V \quad (88)$$
This is actually a special case of the **Virial theorem**. We interpret virial theorem for our discussion as: If the orbit of the particle under a central force is bound and periodic (this is not a necessary condition, but for simplicity we assume this is true), then we will always have

\[
< T > = -\frac{1}{2} < \frac{\partial V}{\partial r} r >
\]  

(89)

where the expression \(< \ldots >\) means the mean of the quantity. If the potential \(V\) is a power-law function of \(r\), i.e. \(V = ar^{n+1}\), we will have

\[
< T > = -\frac{1}{2} \left( n + 1 \right) < V >
\]  

(90)

Furthermore, if the central force is an inverse square law, i.e. \(V = ar^{-1}\) (or \(n = 0\)), we have

\[
< T > = -\frac{1}{2} < V >
\]  

(91)

which is the result we obtained here.

If readers wish to learn more about Virial theorem, they may refer to *Goldstein* section 3.4.

---

- **(1 pt)**: Reducing the Yukawa potential to \(V = -\frac{k}{r}\).
- **(2 pt)**: Solving \(r_c\) from the equation of motion for \(r\).
- **(1 pt)**: Correct expression for \(f\).
- **(1 pt)**: Correct expression for \(E\).

**Total sub-points : 5**

(d) For the Yukawa potential, give the equation whose solution yields the radius of the circular orbit \(r_{c,a}\) in terms of \(\ell, k, m\) and \(a\).

**[Solution]**

For the Yukawa potential, the equation of motion for \(r\) is

\[
m\ddot{r} - \frac{l^2}{mr^3} + \frac{\partial V(r)}{\partial r} = 0
\]  

(92)

\[
m\ddot{r} - \frac{l^2}{mr^3} + \frac{\partial}{\partial r} \left( -\frac{k}{r} \exp \left( -\frac{r}{a} \right) \right) = 0
\]  

(93)

\[
m\ddot{r} - \frac{l^2}{mr^3} + \frac{k}{r^2} \exp \left( -\frac{r}{a} \right) + \frac{k}{ar} \exp \left( -\frac{r}{a} \right) = 0
\]  

(94)

For circular orbit, we have \(r = r_{c,a}\) =Constant and \(\dot{r} = \ddot{r} = 0\), so the equation of motion for \(r\) becomes

\[- \frac{l^2}{mr^3_{c,a}} + \frac{k}{r_{c,a}^2} \exp \left( -\frac{r_{c,a}}{a} \right) + \frac{k}{ar_{c,a}} \exp \left( -\frac{r_{c,a}}{a} \right) = 0
\]  

(95)

\[
\frac{l^2}{m} = kr_{c,a} \left( 1 + \frac{r_{c,a}}{a} \right) \exp \left( -\frac{r_{c,a}}{a} \right)
\]  

(96)

Solving the above equation will give the radius of the circular orbit \(r_{c,a}\).

- **(1 pt)**: Putting in the Yukawa potential into the equation of motion for \(r\). Full credit should also be given.
- **(2 pt)**: Implementing \(r = r_{c,a}\) and \(\dot{r} = \ddot{r} = 0\) into the equation, yielding the correct final form of the equation. Note that students may be have the same form of equation as the solution. As long as they have implemented the conditions stated, credit should be given. Also note that students may also approach the problem through evaluating \(\frac{\partial V_{eff}}{\partial r} |_{r=r_{c,a}}\). Full credit should also be given.
In the limit $r_c/a \ll 1$, solve the equation from part (d) by approximation, to give $r_{c,a}$ in terms of $r_c$ and $a$. You can assume $r_{c,a} = r_c g$, where $g$ is a function of $r_c/a$ (and which should go to 1 in the limit $a \to \infty$). Find $g$ to leading order in $r_c/a$. Also find the orbital frequency in this case.

**[Solution]**

Students are reminded that $r_c = \frac{l^2}{mk}$ is not equal to $r_{c,a}$.

We have the equation as

$$\frac{l^2}{m} = kr_{c,a}(1 + \frac{r_{c,a}}{a}) \exp\left(-\frac{r_{c,a}}{a}\right) \quad (97)$$

$$kr_c = kr_{c,a}(1 + \frac{r_{c,a}}{a}) \exp\left(-\frac{r_{c,a}}{a}\right) \quad (98)$$

$$r_c = r_{c,a}(1 + \frac{r_{c,a}}{a}) \exp\left(-\frac{r_{c,a}}{a}\right) \quad (99)$$

If we assume $r_{c,a} = r_c g$, we have the equation as

$$r_c = gr_c\left(1 + \frac{gr_c}{a}\right) \exp\left(-\frac{gr_c}{a}\right) \quad (100)$$

$$1 = g\left(1 + \frac{gr_c}{a}\right) \exp\left(-\frac{gr_c}{a}\right) \quad (101)$$

In the regime that $\frac{r_c}{a} \ll 1$, we have

$$1 = g\left(1 + \frac{gr_c}{a}\right)\left(1 - \frac{gr_c}{a} + \frac{g^2 r_c^2}{2a^2} + \mathcal{O}\left(\frac{g^3 r_c^3}{a^3}\right)\right) \quad (102)$$

$$1 \approx g\left(1 + \frac{gr_c}{a}\right)\left(1 - \frac{gr_c}{a} + \frac{g^2 r_c^2}{2a^2}\right) \quad (103)$$

$$1 = g\left(1 - \frac{g^2 r_c^2}{2a^2}\right) + \mathcal{O}\left(\frac{g^3 r_c^3}{a^3}\right) \quad (104)$$

$$1 \approx g\left(1 - \frac{g^2 r_c^2}{2a^2}\right) \quad (105)$$

$$g = \left(1 - \frac{g^2 r_c^2}{2a^2}\right)^{-1} \quad (106)$$

$$g = \left(1 + \frac{g^2 r_c^2}{2a^2} + \mathcal{O}\left(\frac{g^4 r_c^4}{4a^4}\right)\right) \quad (107)$$

$$g \approx 1 + \frac{g^2 r_c^2}{2a^2} \quad (108)$$

Hence we have

$$\frac{g^2 r_c^2}{2a^2} - g + 1 = 0 \quad (109)$$

$$g = \frac{1 \pm \sqrt{1 - \frac{2r_c^2}{a^2}}}{\frac{r_c^2}{a^2}} \quad (110)$$

$$g = \frac{a^2}{r_c^2} \pm \frac{a^2}{r_c^2} \sqrt{1 - \frac{2r_c^2}{a^2}} \quad (111)$$

$$g = \frac{a^2}{r_c^2} \pm \frac{a^2}{r_c^2} \left(1 - \frac{1}{2} \frac{2r_c^2}{a^2} - \frac{1}{8} \frac{4r_c^4}{a^4} + \mathcal{O}\left(\frac{8r_c^6}{a^6}\right)\right) \quad (112)$$
\[ g \approx \frac{a^2}{r_c^2} \pm \frac{a^2}{r_c^2} \left( 1 - \frac{r_c^2}{a^2} - \frac{r_c^4}{2a^4} \right) \]  
(113)

\[ g = 1 + \frac{r_c^2}{2a^2} \quad \text{or} \quad \frac{2a^2}{r_c^2} - 1 - \frac{r_c^2}{2a^2} \quad \text{(rejected)} \]  
(114)

Hence, we have

\[ r_{c,a} = r_c g = r_c \left( 1 + \frac{r_c^2}{2a^2} \right) \]  
(115)

Again, recall that the angular momentum is given by

\[ l = mr^2 \dot{\theta} = mr^2 \omega \]  
(116)

where

\[ \omega = \frac{l}{mr^2} = \frac{l}{mr_{c,a}^2} \]  
(117)

is the orbital angular velocity. From this, the orbital frequency is given by

\[ f = \frac{\omega}{2\pi} = \frac{l}{2\pi mr_{c,a}^2} = \frac{l}{2\pi mr_{c,a}^2 \left( 1 + \frac{r_c^2}{2a^2} \right)^2} \]  
(118)

\begin{itemize}
  \item \textbf{(1 pt)}: Replacing \( l^2 \) by \( mkr_c \), and substituting \( r_{c,a} \) by \( gr_c \) in the equation obtained in part (d).
  \item \textbf{(1 pt)}: Expanding \( \exp(-gr_c/a) \) up to the second order term. If students keep only up to the first order term, award only 0.5 point.
  \item \textbf{(2 pt)}: Simplifying the equation, and keeping terms only up to the second order of \( gr_c/a \).
  \item \textbf{(2 pt)}: Solving for \( g \), keeping terms only up to the second order of \( gr_c/a \). Note that students should not keep the negative solution for \( g \) since we require \( g \to 1 \) when \( a \to \infty \).
  \item \textbf{(2 pt)}: Obtaining the correct expression for the orbital frequency in terms of \( l \) and \( r_c \).
\end{itemize}

\textbf{Total sub-points : 8}

(f) If the orbit is nearly circular, i.e. \( r(t) = r_{c,a} + \delta r(t) \) (\( \delta r(t) \ll r_0 \)), derive the differential equation for \( \dot{\delta r}(t) \). Show that it is stable, and corresponds to oscillations about the circular orbit, and find the frequency of oscillation. compare to the orbital frequency.

\begin{itemize}
  \item \textbf{Solution}
    
    Note that the equation of motion for \( r \) is given by

    \[ m \ddot{r} = - \frac{\partial V_{\text{eff}}}{\partial r} \]  
    (119)

    In part (c) and (d), we evaluate the above equation at \( r = r_c \) or \( r = r_{c,a} \) such that either terms in the above equation is 0 (since the radius is fixed, \( \ddot{r} = 0 \); also, \( V_{\text{eff}} \) is minimized at \( r = r_c \) or \( r = r_{c,a} \), and hence its first derivative evaluated at these radii will vanish).

    Now, for a slightly non-circular orbit, its radius is given by

    \[ r(t) = r_{c,a} + \delta r. \]  
    (120)

    We can also evaluate the equation of motion for \( r \) at this radius. The left hand side is simply

    \[ m \ddot{r} = m \frac{d^2}{dt^2} (r_{c,a} + \delta r) = m \ddot{r} \]  
    (121)

    The right hand side becomes

    \[ \frac{\partial V_{\text{eff}}}{\partial r} \bigg|_{r=r_{c,a}+\delta r} \]  
    (122)

\end{itemize}
We can approximate this term by doing a Taylor expansion about \( r = r_{c,a} \) as

\[
\begin{align*}
\frac{\partial V_{\text{eff}}}{\partial r} \bigg|_{r=r_{c,a}+\delta r} & \approx \left. \frac{\partial V_{\text{eff}}}{\partial r} \right|_{r=r_{c,a}} + \left. \frac{\partial^2 V_{\text{eff}}}{\partial r^2} \delta r \right|_{r=r_{c,a}} \\
& = \left. \frac{\partial}{\partial r} \left[ -\frac{l^2}{mr^3} + \frac{k}{r^2} \exp \left( -\frac{r}{a} \right) + \frac{k}{ar} \exp \left( -\frac{r}{a} \right) \right] \right|_{r=r_{c,a}} \delta r \\
& = \left[ \frac{3l^2}{mr_{c,a}^4} - \frac{2k}{r^3} \exp \left( -\frac{r}{a} \right) - \frac{2k}{ar^2} \exp \left( -\frac{r}{a} \right) - \frac{k}{a^2r} \exp \left( -\frac{r}{a} \right) \right] \delta r \\
& = \left[ \frac{3l^2}{m^2r_{c,a}^4} - \frac{k}{ma^2r_{c,a}^3} \left( 2a^2 + 2ar_{c,a} + r_{c,a}^2 \right) \exp \left( -\frac{r_{c,a}}{a} \right) \right] \delta r
\end{align*}
\]

(123)

Since \( V_{\text{eff}} \) is minimized at \( r = r_{c,a} \), the second derivative of \( V_{\text{eff}} \) evaluated at \( r = r_{c} \) (as well as its neighbouring region) should be \( > 0 \). Hence, we can see that the solution for \( \delta r \) is a stable solution.

Hence, the required equation is

\[
\begin{align*}
m\ddot{r} &= -\left[ \frac{3l^2}{m^2r_{c,a}^4} - \frac{k}{ma^2r_{c,a}^3} \left( 2a^2 + 2ar_{c,a} + r_{c,a}^2 \right) \exp \left( -\frac{r_{c,a}}{a} \right) \right] \delta r \\
\dot{r} &= -\left[ \frac{3l^2}{m^2r_{c,a}^4} - \frac{k}{ma^2r_{c,a}^3} \left( 2a^2 + 2ar_{c,a} + r_{c,a}^2 \right) \exp \left( -\frac{r_{c,a}}{a} \right) \right] \delta r
\end{align*}
\]

(124)

(125)

(126)

From the above equation, we can obtain the “angular velocity” of the changing of the orbital radius as

\[
\begin{align*}
\Omega &= \sqrt{3 \frac{l^2}{m^2r_{c,a}^4} - \frac{k}{ma^2r_{c,a}^3} \left( 2a^2 + 2ar_{c,a} + r_{c,a}^2 \right) \exp \left( -\frac{r_{c,a}}{a} \right) } \\
&= \frac{l}{mr_{c,a}^2} \sqrt{3 - \frac{mk_{c,a}}{a^2l^2} \left( 2a^2 + 2ar_{c,a} + r_{c,a}^2 \right) \exp \left( -\frac{r_{c,a}}{a} \right) } \\
&= \omega \sqrt{3 - \frac{mk_{c,a}}{a^2l^2} \left( 2a^2 + 2ar_{c,a} + r_{c,a}^2 \right) \exp \left( -\frac{r_{c,a}}{a} \right) }
\end{align*}
\]

(127)

(128)

(129)

(130)

(131)

and hence the frequency of oscillation is

\[
F = \frac{\Omega}{2\pi} = f \left( 3 - \frac{mk_{c,a}}{a^2l^2} \left( 2a^2 + 2ar_{c,a} + r_{c,a}^2 \right) \exp \left( -\frac{r_{c,a}}{a} \right) \right)
\]

(132)

- (1 pt): Showing that \( m\ddot{r} = m\dot{\delta r} \).
- (1 pt): Arguing / stating that \( \frac{\partial V_{\text{eff}}}{\partial r} \) is 0 when evaluated at \( r = r_{c,a} \).
- (2 pt): Expanding \( \frac{\partial V_{\text{eff}}}{\partial r} \) around \( r = r_{c,a} \) up to \( \frac{\partial^2 V_{\text{eff}}}{\partial r^2} \). Note that if students uses the result that \( \frac{\partial V_{\text{eff}}}{\partial r} \) is 0 when evaluated at \( r = r_{c,a} \) here directly without stating, the credit for the previous item should still be awarded.
- (1 pt): Arguing that since \( \frac{\partial^2 V_{\text{eff}}}{\partial r^2} < 0 \) around \( r = r_{c,a} \), the solution to \( \delta r \) is stable.
- (1 pt): Obtaining the “angular velocity” from the resulting equation.
- (1 pt): Expressing the frequency of oscillation in terms of the orbital frequency for a circular orbit \( f \). Note that the students’ final expression inside the square root might be different from the solution. As long as they have the “\( f \)” dependence in their solutions, credit should be awarded.

Total sub-points : 7
Think about how you might use observations of planetary motion to measure or limit the value of \(a\). You might also want to think about what physics might result in a Yukawa potential.

**Solution**

Note that the frequency of oscillation of the orbit radius is *NOT EQUAL* to the orbital frequency. This implies that when the particle finishes a complete cycle around the center (focus) (i.e. when \(\theta(T_{\text{orbital}}) = 2\pi\), where \(T_{\text{orbital}}\) is the orbital period), it will not be at the same position as it was initially (i.e. \(r(T_{\text{orbital}} \neq r(0))\)). Such a difference will cause the orbit to move (rotate) in time, a phenomenon known as *apsidal precession*.

Since the orbit is moving (rotating), the perihelion is also moving in time, which we call “advancing”. Perihelion advancement is observed for mercury and other celestial objects. Indeed, physicists have been investigating the deviation of short-range gravitational force from the usual Newtonian gravitational force by adding a assumed-Yukawa-like potential \(1\) to the usual Newtonian potential. By measuring the apsidal precession rate of celestial objects, and comparing with the theoretically calculated results, one can provide somewhat a constraint to the value of \(a\) in the Yukawa potential.

In fact, some physicists also postulated the existence of a “fifth force”, aside from the weak, strong, electromagnetic and gravitational forces, in the form of the Yukawa potential \(2\).

- **(1 pt)**: Arguing / Stating / Explaining that the difference in orbital frequency and frequency of oscillation of the orbital radius will lead to apsidal precession.
- **(1 pt)**: Suggesting that measuring precession rates of orbits and comparing with theoretical results may provide constraints on \(a\), or other reasonable suggestions.

Total sub-points : 2

3. **Precessing ellipses**: Hand and Finch Problem 4-24 (slightly changed). Discuss the motion of a particle in a central force potential

\[ V(r) = -\frac{k}{r} + \frac{\beta}{r^2}. \]

In particular, show that the equation of the orbit has an exact solution in the form

\[ \frac{p}{r} = 1 + \epsilon \cos \alpha \phi. \]

This is an ellipse for \(\alpha = 1\), but it is a precessing ellipse if \(\alpha \neq 1\). The precessing motion may be described in terms of the rate of precession of the apsides (turning points). Derive an approximate expression for the rate of precession when \(\alpha\) is close to 1. Calculate the precession angle for one period of the (almost)

---

1See [https://link.springer.com/article/10.1134/S1063772917050092](https://link.springer.com/article/10.1134/S1063772917050092) for example.

elliptical motion. If $\beta$ is increased to the point where it is no longer small compared to the centrifugal term, how does this affect the orbit?

**[Solution]**

Here, we assume the mass $M$ of the source of the central force is much larger than the particle’s mass $m$. In general, one should replace the $m$ in the following discussion by the reduced mass $\mu$.

As usual, we first write down the Lagrangian of the particle as

$$L = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\phi}^2 + \frac{k}{r} - \frac{\beta}{r^2} \quad (133)$$

Again, since the Lagrangian does not depend explicitly on the coordinate $\theta$, its conjugate momentum (which is the total angular momentum $l$) is a time-conserved quantity. The angular momentum can be found by

$$l = \frac{\partial L}{\partial \dot{\phi}} = m r^2 \dot{\phi} \quad (134)$$

Using the Lagrange equation

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0, \quad (135)$$

the equation of motion for $r$ can be calculated as

$$\frac{d}{dt}(m \dot{r}) - \left( m r^2 \dot{\phi}^2 - \frac{k}{r} + \frac{2\beta}{r^3} \right) = 0 \quad (136)$$

$$m \ddot{r} - \frac{l^2}{mr^3} + \frac{k}{r^2} - \frac{2\beta}{r^3} = 0 \quad (137)$$

Note that

$$l = m r^2 \dot{\phi} = m r^2 \frac{d\phi}{dt} \quad (138)$$

$$dt = \frac{m r^2}{l} d\phi \quad (139)$$

$$\frac{d}{dt} = \frac{l}{m r^2} \frac{d}{d\phi} \quad (140)$$

Hence we can write the equation of motion for $r$ as

$$m \ddot{r} - \frac{l^2}{mr^3} + \frac{k}{r^2} - \frac{2\beta}{r^3} = 0 \quad (141)$$

$$m \frac{d}{dt} \frac{d}{dt}(r) - \frac{l^2}{mr^3} + \frac{k}{r^2} - \frac{2\beta}{r^3} = 0 \quad (142)$$

$$m \left( \frac{l}{mr^2} \frac{d}{d\phi} \right) \left( \frac{l}{mr^2} \frac{d}{d\phi} \right) (r) - \frac{l^2}{mr^3} + \frac{k}{r^2} - \frac{2\beta}{r^3} = 0 \quad (143)$$

$$\frac{l^2}{mr^2} \frac{d}{d\phi} \left( \frac{r'}{r} \right) - \frac{l^2}{mr^3} + \frac{k}{r^2} - \frac{2\beta}{r^3} = 0 \quad (144)$$

$$\frac{l^2}{mr^2} \left( \frac{(r^2)(r'') - (r')(2rr')}{r^4} \right) - \frac{l^2}{mr^3} + \frac{k}{r^2} - \frac{2\beta}{r^3} = 0 \quad (145)$$

$$\frac{l^2}{mr^2} \frac{r''}{r} - \frac{2l^2}{m} \frac{r^2}{r^3} - \frac{l^2}{mr^3} + \frac{k}{r^2} - \frac{2\beta}{r^3} = 0 \quad (146)$$

where the “prime” denotes a differentiation with respect to $\phi$.

If we let $u = \frac{1}{r}$, we have

$$\begin{cases} r' = -\frac{u'}{uu} \\ r'' = \frac{(u^2)(u'') - (2uu')}{u^4} = -\frac{u''}{u^2} + 2\frac{u'^2}{u^4} \end{cases} \quad (147)$$
And hence the equation of motion for \( r \) can be written as

\[
\frac{l^2}{m} \left( \frac{-u''}{u^2} + \frac{2u'^2}{u^3} \right) u^4 - \frac{2l^2}{m} \left( \frac{-u'}{u^2} \right)^2 - \frac{l^2}{m} u^3 + ku^2 - 2\beta u^3 = 0
\] (148)

\[
-\frac{l^2}{m} u^2 u'' + \frac{2l^2}{m} u u' + \frac{2l^2}{m} u^2 u' - \frac{l^2}{m} u^3 + ku^2 - 2\beta u^3 = 0
\] (149)

\[
-\frac{l^2}{m} (u^2 u'' + u^3) + ku^2 - 2\beta u^3 = 0
\] (150)

\[
-\frac{l^2}{m} (u'' + u) + k - 2\beta u = 0
\] (151)

\[
\frac{l^2}{m} u'' + \left( \frac{l^2}{m} + 2\beta \right) u = k
\] (152)

The equation we obtained is a second-order inhomogeneous differential equation in \( u \). First we solve for the homogeneous solution for the equation

\[
\frac{l^2}{m} u'' + \left( \frac{l^2}{m} + 2\beta \right) u = 0
\] (153)

by using the trial solution \( u = e^{s\phi} \). We have

\[
\frac{l^2}{m} s^2 + \left( \frac{l^2}{m} + 2\beta \right) = 0
\] (154)

\[
s^2 = -\left( 1 + \frac{2\beta m}{l^2} \right)
\] (155)

\[
s = \pm i \sqrt{1 + \frac{2\beta m}{l^2}}
\] (156)

which is imaginary. This corresponds to sinusoidal functions. Therefore, the homogeneous solution can be written as

\[
u_h = A \cos \left( \sqrt{1 + \frac{2\beta m}{l^2}} \phi + \gamma \right)
\] (157)

where \( A \) and \( \gamma \) are constants to be determined by initial conditions.

Now, we observe that the right hand side of the inhomogeneous equation is a constant \( k \). We are hence prompted to guess a particular solution in the form

\[
u_p = C = \text{constant}
\] (158)

Substituting \( u_p \) into the differential equation, we get

\[
\left( \frac{l^2}{m} + 2\beta \right) C = k
\] (159)

\[
\frac{2\beta m + l^2}{m} C = k
\] (160)

\[
C = \frac{km}{2\beta m + l^2}
\] (161)

Hence, the general solution can be written as

\[
u = u_h + u_p = A \cos \left( \sqrt{1 + \frac{2\beta m}{l^2}} \phi + \gamma \right) + \frac{km}{\frac{2\beta m + l^2}{l}}
\] (162)

\[
= A \cos \left( \frac{\alpha \phi + \gamma}{\alpha^2 l^2} \right) + \frac{km}{\alpha^2 l^2}
\] (163)
where we have denoted

$$\sqrt{1 + \frac{2\beta m}{l^2}} = \alpha$$  \hspace{1cm} (164)

If we assume that the initial change in \( u \) with respect to \( \phi \) is 0, i.e.

$$\left. \frac{\partial u}{\partial \phi} \right|_{\phi=0} = 0$$  \hspace{1cm} (165)

Then we have

$$0 = -A\alpha \sin(\gamma)$$  \hspace{1cm} (166)

$$\gamma = 0$$  \hspace{1cm} (167)

for non-trivial solution. Hence, we can write

$$u = \frac{1}{r} = A \cos(\alpha \phi) + \frac{km}{\alpha^2 l^2}$$  \hspace{1cm} (168)

$$\frac{\alpha^2 l^2}{km} \frac{1}{r} = A\alpha^2 l^2 \cos(\alpha \phi) + 1$$  \hspace{1cm} (169)

$$\frac{p}{r} = 1 + \epsilon \cos(\alpha \phi)$$  \hspace{1cm} (170)

Let’s consider the apsides (the perihelion) of a sinusoidal function. We have, from the previous equation, that

$$r = \frac{p}{1 + \epsilon \cos(\alpha \phi)}$$  \hspace{1cm} (171)

Apsides are positions when the particle is furthest away from the “center” (focus), i.e. \( r \) is maximized. We can easily see that the maximum \( r \) happens when the denominator of the above equation is the smallest, i.e. \( \cos(\alpha \phi) = -1 \),

which corresponds to

$$\alpha \phi = (2n + 1)\pi \quad (n=0, 1, 2...)$$  \hspace{1cm} (173)

Hence, the positions of the apsides are given by

$$\phi(n) = \frac{2n + 1}{\alpha} \pi$$  \hspace{1cm} (174)

The precession rate is then given by

$$\delta \phi = \phi(n+1) - \phi(n) = \frac{2(n + 1) + 1}{\alpha} \pi - \frac{2n + 1}{\alpha} \pi = \frac{2\pi}{\alpha}$$  \hspace{1cm} (175)

When \( \alpha \to 1 \), we have

$$\frac{2\beta m}{l^2} \to 0$$  \hspace{1cm} (176)

and hence the precession rate can be approximated as

$$\delta \phi = 2\pi \left(1 + \frac{2\beta m}{l^2}\right)^{-\frac{1}{2}}$$  \hspace{1cm} (177)

$$= 2\pi \left(1 - \frac{1}{2} \frac{2\beta m}{l^2} + \mathcal{O}\left(\frac{4\beta^2 m^2}{l^4}\right)\right)$$  \hspace{1cm} (178)

$$\approx 2\pi \left(1 - \frac{\beta m}{l^2}\right) = -\frac{2\beta m}{l^2} \pi \quad \text{(mod 2\pi)}$$  \hspace{1cm} (179)
When $\beta$ is no longer small, i.e. $\alpha$ is no longer close to 1, then the precession rate $\delta \phi \pmod{2\pi}$ will not be regular anymore. The figures below show the plot of $r(\phi)$ with various value of $\alpha$.

- (2 pt): Writing down the equation of motion for $r$ in terms of $r$, $l$, $k$ and $\beta$.
- (2 pt): Rewrite the equation of motion in terms of $r'$.
- (4 pt): Solving $r$ in terms of $\theta$.
- (2 pt): Identifying $\alpha$ and $\epsilon$ in the solution for $r$.
- (2 pt): Correct derivation and expression for the position of the apsidies.
- (2 pt): Correct precession rate for $\alpha \to 1$.
- (1 pt): Explaining what will happen if $\beta$ is no longer small.

Total sub-points : 15

4. **Hyperbolic orbits - asymptotes and impact parameters** Hand and Finch Problem 4-27.

A two-body system with reduced mass $\mu$ and orbital angular momentum $l$ in a potential $V(r) = \pm k/r$ with $k > 0$, with eccentricity $\epsilon > 1$, the (unbound) orbit is described in Cartesian coordinates as:

$$p = -\sqrt{x^2 + y^2} + \epsilon x,$$

where $p \equiv l^2/(\mu k)$ is fixed by the initial conditions. The asymptotes (for both attractive and repulsive forces) are straight lines.

(a) Prove that the equations for those lines in Cartesian coordinates for a repulsive force will have the general form

$$y = \pm \left(\sqrt{\epsilon^2 - 1}\right) x - \frac{\epsilon p}{\sqrt{\epsilon^2 - 1}}.$$

The asymptotes for an attractive inverse-square force obey a similar equation, but with $\epsilon \to -\epsilon$.

**[Solution]**

We start from the orbit equation:

$$p = -\sqrt{x^2 + y^2} + \epsilon x \quad (180)$$

$$x^2 + y^2 = (\epsilon x - p)^2 \quad (181)$$

$$x^2 + y^2 = \epsilon^2 x^2 - 2\epsilon px + p^2 \quad (182)$$

$$[(\epsilon^2 - 1)x^2 - 2\epsilon px] - y^2 = -p^2 \quad (183)$$

$$\left(x^2 - \frac{2\epsilon p}{\epsilon^2 - 1} x\right) - \frac{y^2}{\epsilon^2 - 1} = -\frac{p^2}{\epsilon^2 - 1} \quad (184)$$

$$\left(x - \frac{\epsilon p}{\epsilon^2 - 1}\right)^2 - \frac{p^2\epsilon^2}{(\epsilon^2 - 1)^2} - \frac{y^2}{\epsilon^2 - 1} = -\frac{p^2}{\epsilon^2 - 1} \quad (185)$$

$$\left(x - \frac{\epsilon p}{\epsilon^2 - 1}\right)^2 - \frac{y^2}{\epsilon^2 - 1} = \frac{p^2}{(\epsilon^2 - 1)^2} \quad (186)$$

$$\left(x - \frac{\epsilon p}{\epsilon^2 - 1}\right)^2 - \frac{y^2}{\epsilon^2 - 1} = 1 \quad (187)$$
Comparing to the hyperbola equation
\[
\frac{(x - x_0)^2}{a^2} - \frac{(y - y_0)^2}{b^2} = 1, \tag{188}
\]
we see that the “center” of the hyperbolas is
\[
(x, y) = \left( \frac{p \epsilon}{\epsilon^2 - 1}, 0 \right) \tag{189}
\]
The asymptotes of hyperbolas are straight lines passing through the center. To find the slope(s) of the asymptotes, first note that the orbit equation can be written in terms of polar coordinates as
\[
p = -r + \epsilon r \cos \theta \tag{190}
\]
\[
\frac{p}{r} = -1 + \epsilon \cos \theta \tag{191}
\]
As \(r \to \infty\), we have
\[
0 = -1 + \epsilon \cos \theta \tag{192}
\]
\[
\cos \theta = \frac{1}{\epsilon} \tag{193}
\]
\[
\sec \theta = \epsilon \tag{194}
\]
\[
\sec^2 \theta = \epsilon^2 \tag{195}
\]
\[
1 + \tan^2 \theta = \epsilon^2 \tag{196}
\]
\[
\tan^2 \theta = \epsilon^2 - 1 \tag{197}
\]
\[
\tan \theta = \pm \sqrt{\epsilon^2 - 1} \tag{198}
\]
Hence, the equation of asymptotes are
\[
\frac{y - 0}{x - \frac{p \epsilon}{\epsilon^2 - 1}} = \pm \sqrt{\epsilon^2 - 1} \tag{199}
\]
\[
y = \pm \sqrt{\epsilon^2 - 1} x \mp \frac{p \epsilon}{\sqrt{\epsilon^2 - 1}} \tag{200}
\]
• (1 pt) : Deriving the slope of the asymptotes.
• (2 pt) : Deriving the center of the hyperbolas.
• (1 pt) : Deriving the equations of the asymptotes.

Total sub-points : 4

(b) The impact parameter \(b\) is defined as the distance of closest approach to the origin along the incoming asymptote. Prove that the angular momentum \(l\) is given in terms of the reduced mass \(\mu\) and the center of mass relative velocity \(v_\infty\) infinitely far away from the origin by the formula \(l = \mu v_\infty b\).

[Solution]
As we have seen for many times, since under the proposed central force, the Lagrangian does not depend explicitly on the angle \(\theta\), its conjugate momentum, which is the angular momentum, is a time-conserved quantity. Hence, wherever we evaluate the angular momentum does not affect its value, so we can calculate its angular momentum at infinity, which is simply given by
\[
l = \mu v_\infty b. \tag{201}
\]
• (1 pt) : Arguing that the angular momentum is a conserved quantity.
• (1 pt) : Explaining why the angular momentum can be given by the expression suggested in the question.
(c) Show that the impact parameter can be written as a function of $\epsilon$ for constant total energy in the center of mass, $E$:

$$b(\epsilon) = \frac{k}{2E} \sqrt{\epsilon^2 - 1}. $$

[Solution]

Initially, the particle is at infinity and hence the only energy in the system is the kinetic energy of the particle. The energy at any position $r$ is given by

$$E = \frac{1}{2} \mu \dot{r}^2 - \frac{l^2}{2mr^2} \pm \frac{k}{r}. \quad (202)$$

Since energy is conserved, we have

$$E = \frac{1}{2} \mu v^2 \quad (203)$$

$$v_\infty = \sqrt{\frac{2E}{\mu}}. \quad (204)$$

Hence, the angular momentum can be written as

$$l = \mu v_\infty b = \mu b \sqrt{\frac{2E}{\mu}} = b \sqrt{2\mu E}. \quad (205)$$

Now, we wish to find $b$ in terms of $p$ and $\epsilon$. Recall that $b$ is the closest approach to the origin along the incoming asymptote. We find the shortest distance from the origin $(0,0)$ to the asymptote. Any point on the asymptote can be denoted by $(x, \sqrt{\epsilon^2 - 1}x - \frac{\epsilon p}{\sqrt{\epsilon^2 - 1}})$. The distance $s$ between the origin and the line is given by

$$s = \sqrt{x^2 + \left(\sqrt{\epsilon^2 - 1}x - \frac{\epsilon p}{\sqrt{\epsilon^2 - 1}}\right)^2} = \sqrt{\epsilon^2 x^2 - 2\epsilon px + \frac{\epsilon^2 p^2}{\epsilon^2 - 1}} = \sqrt{f(x)} \quad (206)$$

To find the closest distance, we simply have to find the minimum value of $f(x)$, i.e.

$$\frac{df}{dx} = 0 \quad (207)$$

$$2\epsilon^2 x - 2\epsilon p = 0x = \frac{p}{\epsilon} \quad (208)$$

Putting this value of $x$ back into $s$, we will obtain $b$ as

$$s = b = \sqrt{\epsilon^2 \left(\frac{p}{\epsilon}\right)^2 - 2\epsilon p \left(\frac{p}{\epsilon}\right) + \frac{\epsilon^2 p^2}{\epsilon^2 - 1}} \quad (209)$$

$$= \sqrt{\frac{p^2}{\epsilon^2 - 1}} \quad (210)$$

$$= \frac{p}{\epsilon} \sqrt{\frac{1}{\epsilon^2 - 1}} \quad (211)$$

$$= \frac{l^2}{\mu k} \sqrt{\frac{1}{\epsilon^2 - 1}} \quad (212)$$

$$= \frac{l^2}{\mu k} \sqrt{\frac{1}{\epsilon^2 - 1}} \quad (213)$$
With \( l = b\sqrt{2\mu E} \), we have

\[
\begin{align*}
b &= \frac{(b\sqrt{2\mu E})^2}{\mu k} \sqrt{\frac{1}{\epsilon^2 - 1}} \quad (214) \\
b &= \frac{2b^2 E}{k} \sqrt{\frac{1}{\epsilon^2 - 1}} \quad (215) \\
b &= \frac{k}{2E} \sqrt{\epsilon^2 - 1} \quad (216)
\end{align*}
\]

- (2 pt): Expressing \( v_\infty \) in terms of \( E \) and \( \mu \).
- (3 pt): Expressing \( b \) in terms of \( l, \mu \) and \( \epsilon \).
- (2 pt): Deriving the final expression.

Total sub-points : 7

Maximum points obtainable for this problem set: 81 pt