Physics 106a: Assignment 4 (Solutions prepared by Alvin Li)

1 November, 2019; due 7pm Friday, 8 November in the "Ph106 In Box" in East Bridge mailbox.

Oscillations and Normal Modes

Reading

Hand and Finch Chapters 3 and 9.

Problems

1. **Driven critically damped oscillator:** (This is Hand and Finch Problem 3-21). A critically damped oscillator has $Q = 1/2$. The free oscillator obeys the homogeneous equation of motion $\ddot{q} + 2\dot{q} + q = 0$ (in natural units where $\omega_0 = 1$, or equivalently, using rescaled dimensionless time $\omega_0 t \to t$). The two free oscillator solutions are e^{-t} and te^{-t}.

   (a) Drive this oscillator with an external driving force that is a discontinuous step at $t = 0$: for $t < 0$, $F = 0$, and for $t \geq 0$, $F = 1$. Assuming that $q = \dot{q} = 0$ for $t < 0$, find an explicit solution for $t \geq 0$.

   [Solution]

   The force can be described by a step-function, i.e.

$$F(t) = \theta(t) = \begin{cases} 0 & (t < 0) \\ 1 & (t \geq 0) \end{cases} \tag{1}$$

   For $t < 0$, the solution is simply given by the linear combination of the homogeneous solutions,

$$q(t < 0) = Ae^{-t} + Bte^{-t}, \tag{2}$$

   where A and B are undetermined constants.

   Yet, with the initial conditions $q(t) = \dot{q}(t) = 0$ for $t < 0$, we have $A = B = 0$ and hence

$$q(t < 0) = 0. \tag{3}$$

   For $t \geq 0$, we have the equation of motion as

$$\ddot{q} + 2\dot{q} + q = 1. \tag{4}$$

   The particular solution can be guessed to be in the form

$$q_p = C = \text{Constant}. \tag{5}$$

   Substituting the guessed solution into 4, we have

$$C = 1, \tag{6}$$

   and so the general solution for $t \geq 0$ is

$$q(t) = q_p + q_c = 1 + Me^{-t} + Nte^{-t}, \tag{7}$$

   where M and N are undetermined constants.

   With the initial conditions, we have

(1) For $q(0) = 0$:

$$M + 1 = 0 \tag{8}$$
$$M = -1 \tag{9}$$

(2) For $\dot{q} = 0$:

$$\dot{q} = e^{-t} + Ne^{-t} - Nte^{-t} = 0 \tag{10}$$
$$1 + N = 0 \tag{11}$$
$$N = -1 \tag{12}$$

Hence the solution is

$$q(t) = \begin{cases} 0 & (t < 0) \\ 1 - e^{-t} - te^{-t} & (t \geq 0) \end{cases} \tag{13}$$

- **(1 pt)** : Identifying the applied force as a step-function.
- **(1 pt)** : A correct guess for the form of the particular solution. If the student jumps to the solution immediately, this point should also be awarded.
- **(1 pt)** : Solving the particular solution for $q \geq 0$.
  [The follow points should only be awarded if students had also considered the homogeneous solutions as part of the general solution.]
- **(1 pt)** : Solving for M in the general solution.
- **(1 pt)** : Solving for N in the general solution.

Total sub-points : 5

(b) Consider the oscillatory driving force: for $t < 0$, $F = 0$, and for $t \geq 0$, $F = \cos t$. Again, $q = \dot{q} = 0$ for $t < 0$. Find the form of the steady-state ($t \gg 1$) solution by first solving for $q(t)$ for the complex driving force $F = e^{it}$, $t > 0$, and then finding the physical displacement of the oscillator $q(t)$ for $F(t) = \cos t$. What is the relative phase between the driving force and the oscillator response in the steady state?

[Solution]

We write the force as a complex function, i.e.

$$F(t) = e^{it}, \tag{14}$$

and the equation of motion can be written as

$$\ddot{\tilde{q}} + 2\dot{\tilde{q}} + \tilde{q} = e^{it}, \tag{15}$$

where the tilde means that $\tilde{q}$ can be in general complex.
The particular solution can be guessed to be in the form

$$q_p = \tilde{C}e^{it}. \tag{16}$$

Substituting the guessed solution into 15, we have

$$-\tilde{C} + 2i\tilde{C} + \tilde{C} = 1 \tag{17}$$
$$\tilde{C} = \frac{1}{2i} = -\frac{i}{2} = \frac{1}{2}e^{-\frac{i\pi}{2}} \tag{18}$$

and so the particular solution for is

$$\tilde{q}(t) = \frac{1}{2}e^{-\frac{i\pi}{2}}e^{it} = \frac{1}{2}e^{i\left(t-\frac{\pi}{2}\right)}, \tag{19}$$

which, in this form, it is easy for us to see that the *relative phase* between the applied force and the response is

$$|\Delta\phi| = \frac{\pi}{2}. \tag{20}$$

Hence, the steady-state solution is simply given by

$$q_p(t) = \Re(\tilde{q}) = \frac{1}{2}\cos\left(t - \frac{\pi}{2}\right) \tag{21}$$

Supplementary box 4.1: Transient solution and steady-state solution

For forced damped oscillations, we can classified the solution into *transient* and *steady-state* solution as

$$q = \underbrace{q_p}_{\text{steady-state}} + \underbrace{q_h}_{\text{transient}}, \tag{22}$$

that is, the particular solution is the *steady-state* solution and the homogeneous solution is the *transient* solution.

For example, for the solution in this question, we have

$$q = \underbrace{\frac{1}{2}\cos\left(t - \frac{\pi}{2}\right)}_{\text{steady-state}} + \underbrace{Ae^{-t} + Bte^{-t}}_{\text{transient}}. \tag{23}$$

Let's look at the transient solution:

$$q_{\text{transient}} = Ae^{-t} + Bte^{-t}. \tag{24}$$

This part carries information about the **initial conditions** of the oscillator. However, since both terms carry an *exponential decaying* part, they will eventually *die away* as time goes on. This means that the initial information will be lost after a while, and that's why this part of the solution as named as *transient*.

On the other hand, the steady-state solution

$$q_{\text{steady-state}} = \frac{1}{2}\cos\left(t - \frac{\pi}{2}\right) \tag{25}$$

is an oscillatory function which will not die away even as time goes (i.e. it will remain here forever). Hence it is granted the name *steady-state solution*. It carries no information of the initial conditions of the oscillator, but simply the response of it to the applied force (and damping).

- **(1 pt)** : A correct guess for the form of the complex particular solution. If the student solve directly with a real force, i.e. $F = \cos t$, *DO NOT* award any points.
- **(1 pt)** : Solving the complex particular solution.
- **(1 pt)** : Correct relative phase between the response and applied force.
- **(1 pt)** : Writing down the real (physical) particular solution.

Total sub-points : 4

(c) To find the exact solution for all positive times you could use a Green's function or you could match boundary conditions at $t = 0$. Use the boundary condition method to find the transient solution. Combine this with the result of part (b) to find the oscillator's total response to suddenly turning on a $\cos t$ driving force at $t = 0$. Make a sketch of $q(t)$ for $0 \le t \le 4$. For what time is the response maximized?

[Solution]

The general solution can be written as

$$q = q_p + q_h = \frac{1}{2}\cos\left(t - \frac{\pi}{2}\right) + Ae^{-t} + Bte^{-t}, \tag{26}$$

where A and B are undetermined constants.

Using the initial conditions, we have

(1) For $q(0) = 0$:

$$\frac{1}{2}\cos\left(0 - \frac{\pi}{2}\right) + A = 0 \tag{27}$$

$$A = 0 \tag{28}$$

(2) For $\dot{q} = 0$:

$$\dot{q} = -\frac{1}{2}\sin\left(t - \frac{\pi}{2}\right) + Be^{-t} - Bte^{-t} = 0 \tag{29}$$

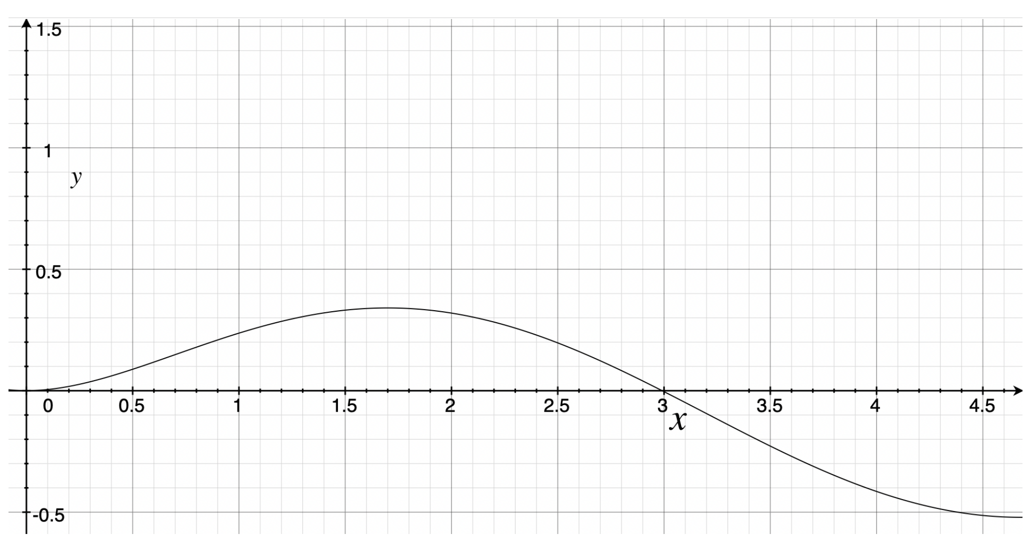
$$\frac{1}{2} + B = 0 \tag{30}$$

$$B = -\frac{1}{2} \tag{31}$$

Hence the solution is

$$q(t) = \frac{1}{2}\cos\left(t - \frac{\pi}{2}\right) - \frac{1}{2}te^{-t} = \frac{1}{2}\sin t - \frac{1}{2}te^{-t} \tag{32}$$

Here shows a graph of $q(t)$ against t:



To find the maximum response, we need to solve

$$\dot{q} = \frac{1}{2}\cos t - \frac{1}{2}e^{-t} + \frac{1}{2}te^{-t} = 0 \tag{33}$$

Using *Wolfram Alpha*, *Mathematica* or other software, one can find that the solutions are $t = 0, 1.70, 4.68...$ Together with the graph, one can find that the maximum response happens at $t = 4.68$, with a corresponding response of $|q| = 0.521$.

- **(1 pt)** : Solving for A in the general solution.
- **(1 pt)** : Solving for B in the general solution
- **(1 pt)** : Graphing $q(t)$ from $t = 0$ to $t = 4$.
- **(1 pt)** : Writing down the equation ($\dot{q} = 0$) which gives local extremum values of q.
- **(1 pt)** : Solving t that gives the maximum response.

Total sub-points : 5

(d) The derivative of a step function is a delta function. From this fact, find the response of this oscillator to a delta function impulse at $t = 0$. Then find the explicit form of the Green's function $G(t - t')$. Write the oscillator response to the driving force in part (b), as an integral over t'. What are the limits of integration? It is easy enough to do the Green's function integral, e.g. using Mathematica, so you might want to do this, for no credit, to check the result in part (c).

[Solution]

Consider the equation of motion for a step-function force,

$$\ddot{q} + 2\dot{q} + q = \theta(t) \tag{34}$$

Differentiating both sides with respect to time t, and write $Q(t) = \dot{q}$, we have

$$\ddot{Q} + 2\dot{Q} + Q = \delta(t). \tag{35}$$

This shows that $Q(t) = \dot{q}(t)$ is clearly a solution to a *unit-impulse force*, i.e.

$$Q(t) = \dot{q}(t) = \frac{d}{dt}\Big(- e^{-t} - te^{-t}\Big) = e^{-t} - e^{-t} + te^{-t} = te^{-t} \tag{36}$$

By shifting the solution to $t = t'$, we have the Green's function as

$$G(t, t') = (t - t')e^{-(t-t')}. \tag{37}$$

For the driving force in question 3(b), the response is simply

$$q(t) = \int_{-\infty}^{t} G(t, t')F(t')dt' = \underbrace{\int_{0}^{t} (t - t')e^{-t+t'} \cos t' dt'}_{\text{starts from } t'=0 \text{ since } q=0 \text{ for } t'<0} \tag{38}$$

Supplementary box 4.2: Doing the integral by hand...
The above integral doesn't seem too difficult indeed. Why don't we try to do it by hand (without using *Mathematica*)? We have the complex solution as

$$\tilde{q}(t) = \int_0^t (t-t')e^{-t+t'}e^{it'}dt' \tag{39}$$
$$= t\int_0^t e^{it'+t'-t}dt' - \int_0^t t'e^{it'+t'-t}dt' \tag{40}$$
$$= t\left[\frac{e^{it'+t'-t}}{i+1}\right]_0^t - \left(\left[\frac{t'e^{it'+t'-t}}{i+1}\right]_0^t - \int_0^t \frac{e^{it'+t'-t}}{i+1}dt'\right) \tag{41}$$
$$= \frac{t(e^{it}-e^{-t})}{i+1} - \frac{te^{it}}{i+1} + \left[\frac{e^{it'+t'-t}}{(i+1)^2}\right]_0^t \tag{42}$$
$$= -\frac{te^{-t}}{i+1} + \frac{e^{it}}{(i+1)^2} - \frac{e^{-t}}{(i+1)^2} \tag{43}$$
$$= -\frac{te^{-t}(1-i)}{2} + \frac{e^{it}(1-i)^2}{4} - \frac{e^{-t}(1-i)^2}{4} \quad \text{(Rationalize the denominator)} \tag{44}$$
$$= -\frac{te^{-t}}{2} + i\frac{te^{-t}}{2} + \frac{e^{it}(1-2i+i^2)}{4} - \frac{e^{-t}(1-2i+i^2)}{4} \tag{45}$$
$$= -\frac{te^{-t}}{2} + i\frac{te^{-t}}{2} + \frac{(\cos t + i\sin t)(-2i)}{4} + i\frac{e^{-t}}{2} \tag{46}$$
$$= -\frac{e^{-t}}{2} + \frac{1}{2}\sin t + \frac{i}{2}(te^{-t} + e^{-t} - \cos t) \tag{47}$$

The physical solution is just the real part of $\tilde{q}$, i.e.

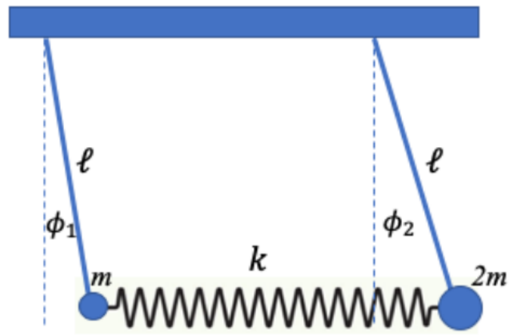
$$q(t) = \Re(\tilde{q}(t)) = \frac{1}{2}\sin t - \frac{e^{-t}}{2} \tag{48}$$

Note that, however, even though we obtained the correct solution here, this is in fact a **complex integral** which should be done in a more careful way. If you are interested, you may go to the library (or Amazon) to look for books on "complex analysis".

- **(1 pt)** : Arguing that the solution to a delta function impulse at $t = 0$ is simply the time derivative of the solution in part (a).
- **(1 pt)** : Correct response for a delta function impulse at $t = 0$.
- **(1 pt)** : Correct expression for the Green's function $G(t - t')$.
- **(1 pt)** : Correct integrand for the oscillator response to the driving force in part (b).
- **(1 pt)** : Correct limits of integration.

Total sub-points : 5

2. **Coupled pendulums:** Two pendulums of identical length l with masses m and $2m$ are connected by a spring of spring constant $k = 2mg/l$. Use the variables ϕ_1 for the lighter mass and ϕ_2 for the heavier mass. The spring is unstretched when the two pendulums are vertical $\phi_1 = \phi_2 = 0$.



(a) What is the Lagrangian up to quadratic order in the regime of small displacement and velocities?

[Solution]

The kinetic energy term is

$$T = \frac{1}{2}ml^2\dot{\phi}_1^2 + \frac{1}{2}(2m)l^2\dot{\phi}_2^2 = \frac{1}{2}ml^2(\dot{\phi}_1^2 + 2\dot{\phi}_2^2). \tag{49}$$

When we calculate the potential energy term (the spring contribution), we assume that the change in length of the spring is solely coming from the *horizontal* displacement of the pendulums (i.e. we ignore the contribution from the vertical displacement). Furthermore, we assume the horizontal displacement can be approximated as

$$\Delta x \approx l\phi_i, \tag{50}$$

which is the arc length of the trajectory traced by each of the pendulums. With these, the potential energy term can be written as

$$\begin{aligned} V &= -mgl\cos\phi_1 - 2mgl\cos\phi_2 + \frac{1}{2}k(l\phi_1 - l\phi_2)^2 & (51)\\ &\approx -mgl + \frac{mgl\phi_1^2}{2} - 2mgl + \frac{2mgl\phi_2^2}{2} + \frac{1}{2}kl^2(\phi_1 - \phi_2)^2 & (52)\\ &= \frac{mgl\phi_1^2}{2} + mgl\phi_2^2 + \frac{1}{2}kl^2(\phi_1 - \phi_2)^2 - 3mgl & (53)\end{aligned}$$

Hence, the Lagrangian is

$$\begin{aligned} \mathcal{L} = T - V &= \frac{1}{2}ml^2(\dot{\phi}_1^2 + 2\dot{\phi}_2^2) - \frac{mgl\phi_1^2}{2} - mgl\phi_2^2 - \frac{1}{2}kl^2(\phi_1 - \phi_2)^2 + 3mgl & (54)\\ &= \frac{1}{2}ml^2(\dot{\phi}_1^2 + 2\dot{\phi}_2^2) - \left(\frac{mgl}{2} + \frac{kl^2}{2}\right)\phi_1^2 + kl^2\phi_1\phi_2 - \left(mgl + \frac{kl^2}{2}\right)\phi_2^2 + 3mgl & (55)\\ &= \frac{1}{2}ml^2(\dot{\phi}_1^2 + 2\dot{\phi}_2^2) - \frac{3mgl}{2}\phi_1^2 + 2mgl\phi_1\phi_2 - 2mgl\phi_2^2 + 3mgl & (56)\end{aligned}$$

- **(1 pt)** : Correct kinetic energy term. Note that if the student does not substitute $k = 2mg/l$, credit should still be given. Same for the potential energy term.

- **(1 pt)** : Correct potential energy term (without approximation).
- **(1 pt)** : Correct approximation to the potential energy term.

Total sub-points : 3

(b) Find the kinetic and potential energy matrices.

[Solution]

The Lagrangian is

$$\mathcal{L} = \frac{1}{2}ml^2(\dot{\phi}_1^2 + 2\dot{\phi}_2^2) - \left(\frac{mgl}{2} + \frac{kl^2}{2}\right)\phi_1^2 + kl^2\phi_1\phi_2 - \left(mgl + \frac{kl^2}{2}\right)\phi_2^2 \tag{57}$$

Hence, the kinetic energy matrix is

$$\mathbf{T} = \begin{bmatrix} \frac{1}{2}ml^2 & 0 \\ 0 & ml^2 \end{bmatrix} \tag{58}$$

and the potential energy matrix is

$$\mathbf{V} = \begin{bmatrix} \frac{mgl}{2} + \frac{kl^2}{2} & -\frac{kl^2}{2} \\ -\frac{kl^2}{2} & +mgl + \frac{kl^2}{2} \end{bmatrix} = \begin{bmatrix} \frac{3mgl}{2} & -mgl \\ -mgl & 2mgl \end{bmatrix} \tag{59}$$

- **(1 pt)** : Expressing the Lagrangian in terms of ϕ_1^2, ϕ_2^2, $\dot{\phi}_1^2$, $\dot{\phi}_2^2$ and $\phi_1\phi_2$. Note that if the student has already done so in part (a), credit should also be given.
- **(1 pt)** : Correct kinetic energy matrix. Again, if the student does not substitute $k = 2mg/l$, credit should still be given.
- **(1 pt)** : Correct potential energy matrix.

Total sub-points : 3

(c) Using these, show that the normal mode frequencies are ω_0 and $2\omega_0$ with $\omega_0 = \sqrt{g/l}$.

[Solution]

To find the normal modes, we need to solve

$$|\mathbf{V} - \omega^2\mathbf{T}| = 0 \tag{60}$$

$$\begin{vmatrix} \frac{mgl}{2} + \frac{kl^2}{2} - \frac{\omega^2}{2}ml^2 & -\frac{kl^2}{2} \\ -\frac{kl^2}{2} & mgl + \frac{kl^2}{2} - \omega^2 ml^2 \end{vmatrix} = 0 \tag{61}$$

After some tedious simplification... (62)

$$2ml^2\omega^4 - (3kl^2 + 4mgl)\omega^2 + (3gkl + 2mg^2) = 0 \tag{63}$$

$$\omega^2 = \frac{(3kl^2+3mgl) \pm \sqrt{(3kl^2+4mgl)^2 - 4(2ml^2)(3gkl+2mg^2)}}{2(2ml^2)} \tag{64}$$

$$\omega^2 = \frac{3kl^2 + 4mgl \pm 3kl^2}{4ml^2} \tag{65}$$

$$\omega^2 = \frac{g}{l} \quad \text{or} \quad \frac{g}{l} + \frac{3k}{2m} = \omega_0^2 \quad \text{or} \quad \omega_0^2 + 3\frac{mg}{l} = \omega_0^2 \quad \text{or} \quad 4\omega_0^2 \tag{66}$$

Hence the normal mode frequencies are given by

$$\omega = \omega_0 \quad \text{or} \quad 2\omega_0 \tag{67}$$

- **(1 pt)** : Writing down the characteristic equation in terms of θ_i and $\dot{\theta}_i$.
- **(2 pt)** : Solving for ω^2 (or ω). Note that students *HAVE TO* substitute $k = 2mg/l$ in this problem in order to show the desired result.
- **(1 pt)** : Expressing ω in terms of ω_0.

Total sub-points : 4

(d) For an initial condition of both pendulum released from rest at $t = 0$, find the ratio of the displacements $\theta_1(0)/\theta_2(0)$ such that only the higher frequency oscillation is excited.

[Solution]

The higher frequency oscillation refers to the oscillation mode with frequency $\omega = 2\omega_0$. We have to look for the normal mode eigenvectors first.

(1) For $\omega = \omega_0 = \sqrt{\frac{g}{l}}$, we have, from the matrix equation $\mathbf{V} - \omega^2 \mathbf{T} = 0$,

$$\begin{cases} \left(\frac{kl^2}{2} + \frac{mgl}{2} - \frac{ml^2}{2} \times \frac{g}{l}\right)\phi_1 - \frac{kl^2}{2}\phi_2 = 0 \\ -\frac{kl^2}{2}\phi_1 + \left(\frac{kl^2}{2} + mgl - ml^2 \times \frac{g}{l}\right)\phi_2 = 0 \end{cases} \tag{68}$$

Solving, we get

$$\phi_1 = \phi_2 \tag{69}$$

which means that the corresponding eigenvector (not normalized) is

$$\vec{\phi}_A = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{70}$$

(2) For $\omega = \sqrt{\omega_0^2 + \frac{3k}{2m}} = \sqrt{\frac{g}{l} + \frac{3k}{2m}} = 2\omega_0$, we have

$$\begin{cases} \left(\frac{kl^2}{2} + \frac{mgl}{2} - \frac{ml^2}{2} \times \left(\frac{g}{l} + \frac{3k}{2m}\right)\right)\phi_1 - \frac{kl^2}{2}\phi_2 = 0 \\ -\frac{kl^2}{2}\phi_1 + \left(\frac{kl^2}{2} + mgl - ml^2 \times \left(\frac{g}{l} + \frac{3k}{2m}\right)\right)\phi_2 = 0 \end{cases} \tag{71}$$

Solving, we get

$$\phi_1 = -2\phi_2 \tag{72}$$

which means that the corresponding eigenvector (not normalized) is

$$\vec{\phi}_B = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \tag{73}$$

Hence, the general solution can be written as

$$\vec{\phi} = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} (A\cos(\omega_0 t) + B\sin(\omega_0 t)) + \begin{bmatrix} -2 \\ 1 \end{bmatrix} (C\cos(2\omega_0 t) + D\sin(2\omega_0 t)) \tag{74}$$

where A, B, C and D are undetermined constants.

The initial conditions $\vec{\theta}(0) = \begin{bmatrix} \phi_1(0) \\ \phi_2(0) \end{bmatrix}$ and $\dot{\vec{\phi}}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ give the set of equations

$$\begin{cases} \phi_1(0) = A - 2C \\ \phi_2(0) = A + C \\ 0 = B\omega_0 - 4D\omega_0 \\ 0 = B\omega_0 + 2D\omega_0 \end{cases} \tag{75}$$

If we want only the higher frequency oscillation mode to be excited, i.e. we want $A = B = 0$, then the equations become

$$\begin{cases} \phi_1(0) = -2C \\ \phi_2(0) = C \\ 0 = -4D\omega_2 \\ 0 = 2D\omega_2 \end{cases} \tag{76}$$

and hence the required ratio is

$$\frac{\phi_1(0)}{\phi_2(0)} = \frac{-2C}{C} = -2 \tag{77}$$

- **(2 pt)** : Solving for the eigenvector corresponding to $\omega = \omega_0$.
- **(2 pt)** : Solving for the eigenvector corresponding to $\omega = 2\omega_0$.
- **(1 pt)** : Writing down the general solution in terms of the eigenvectors. Note that students may also write in the complex solution form, and should also be awarded the credit in this case.
- **(1 pt)** : Implementing the initial conditions. Note that students should still be given the credit if they only implement the condition on the initial position (since this is sufficient), although it is good practice to implement also the condition on the initial velocity.
- **(1 pt)** : Noting that the coefficients corresponding to the normal frequency $\omega 0 = 2\omega_0$ have to be 0.
- **(1 pt)** : Solving the ratio of $\phi_1(0)/\phi_2(0)$.

Total sub-points : 8

(e) A small horizontal force $F = F_0 \cos \omega t$ is applied to the lighter mass. Find an expression for the displacement $\theta_1(t)$ in the driven motion for general ω. You may assume the transient response at the resonant frequencies has died out due to some very small damping.

[Solution]

We return to the original equation of motion:

$$\mathbf{T}\ddot{\vec{\tilde{\phi}}} + \mathbf{V}\vec{\tilde{\phi}} = \vec{F} = \begin{bmatrix} F_0 l e^{i\omega t} \\ 0 \end{bmatrix} \tag{78}$$

We guess the particular solution as

$$\vec{\tilde{\phi}}_p = e^{i\omega t} \begin{bmatrix} \tilde{A} \\ \tilde{B} \end{bmatrix} \tag{79}$$

where the *tilde* means that the constants can be complex.
Substituting into the equation of motion, we have

$$-\omega^2 e^{i\omega t} \begin{bmatrix} \frac{1}{2}ml^2 & 0 \\ 0 & ml^2 \end{bmatrix} \begin{bmatrix} \tilde{A} \\ \tilde{B} \end{bmatrix} + e^{i\omega t} \begin{bmatrix} \frac{mgl}{2} + \frac{kl^2}{2} & -\frac{kl^2}{2} \\ -\frac{kl^2}{2} & +mgl + \frac{kl^2}{2} \end{bmatrix} \begin{bmatrix} \tilde{A} \\ \tilde{B} \end{bmatrix} = \begin{bmatrix} F_0 l e^{i\omega t} \\ 0 \end{bmatrix} \tag{80}$$

$$\underbrace{\begin{bmatrix} -\frac{1}{2}\omega^2 ml^2 + \frac{mgl}{2} + \frac{kl^2}{2} & -\frac{kl^2}{2} \\ -\frac{kl^2}{2} & -\omega^2 ml^2 + mgl + \frac{kl^2}{2} \end{bmatrix}}_{\text{"Monster matrix"}} \begin{bmatrix} \tilde{A} \\ \tilde{B} \end{bmatrix} = \begin{bmatrix} F_0 l \\ 0 \end{bmatrix} \tag{81}$$

To solve for $\tilde{A}$ and $\tilde{B}$, we need to invert the "Monster matrix" and put it to the other side of the equation. The determinant of the monster matrix is given by

$$\begin{aligned}\Delta &= \frac{\omega^4 m^2 l^4}{2} + \frac{m^2 g^2 l^2}{2} - m^2 g l^3 \omega^2 - \frac{3mkl^4\omega^2}{4} + \frac{3mgkl^3}{4} && (82)\\ &= \frac{\omega^4 m^2 l^4}{2} + \frac{m^2 g^2 l^2}{2} - m^2 g l^3 \omega^2 - \frac{3m^2 g l^3 \omega^2}{2} + \frac{3m^2 g^2 l^2}{2} && (83)\\ &= \frac{\omega^4 m^2 l^4}{2} + 2m^2 g^2 l^2 - \frac{5m^2 g l^3 \omega^2}{2} && (84)\end{aligned}$$

and the inverse monster matrix is given by

$$\begin{aligned}&\frac{1}{\Delta}\begin{bmatrix} -\omega^2 m l^2 + mgl + \frac{kl^2}{2} & \frac{kl^2}{2} \\ \frac{kl^2}{2} & -\frac{1}{2}\omega^2 m l^2 + \frac{mgl}{2} + \frac{kl^2}{2} \end{bmatrix} && (85)\\ &= \frac{1}{\Delta}\begin{bmatrix} -\omega^2 m l^2 + 2mgl & mgl \\ mgl & -\frac{1}{2}\omega^2 m l^2 + \frac{3mgl}{2} \end{bmatrix} && (86)\end{aligned}$$

Hence, $\tilde{A}$ and $\tilde{B}$ are given by

$$\begin{aligned}\begin{bmatrix} \tilde{A} \\ \tilde{B} \end{bmatrix} &= \frac{1}{\Delta}\begin{bmatrix} -\omega^2 m l^2 + 2mgl & mgl \\ mgl & -\frac{1}{2}\omega^2 m l^2 + \frac{3mgl}{2} \end{bmatrix}\begin{bmatrix} F_0 l \\ 0 \end{bmatrix} && (87)\\ &= \frac{1}{\Delta}\begin{bmatrix} F_0 l(-\omega^2 m l^2 + 2mgl) \\ F_0 m g l^2 \end{bmatrix} && (88)\end{aligned}$$

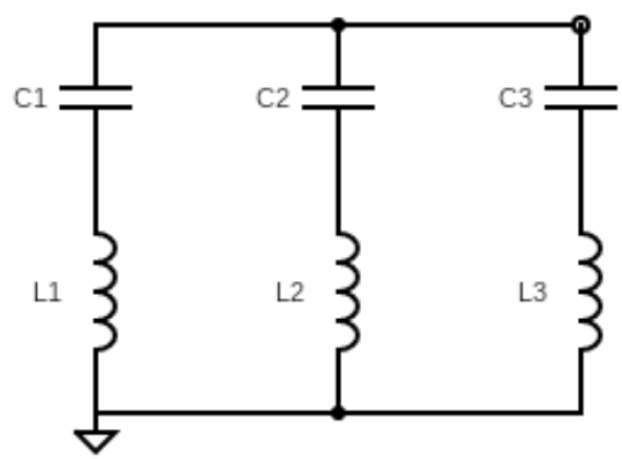
Assuming there is damping such that the transient response dies out, we have the solution as

$$\vec{\phi} = \Re\left(\frac{e^{i\omega t}}{\Delta}\begin{bmatrix} F_0 l(-\omega^2 m l^2 + 2mgl) \\ F_0 m g l^2 \end{bmatrix}\right) = \frac{\cos\omega t}{\Delta}\begin{bmatrix} F_0 l(-\omega^2 m l^2 + 2mgl) \\ F_0 m g l^2 \end{bmatrix} \qquad (89)$$

- **(1 pt)** : Writing down the equation of motion with applied force.
- **(1 pt)** : Correct guess for the particular solution.
- **(4 pt)** : Solving the amplitude A for the guessed particular solution.
- **(1 pt)** : Writing down the final steady-state solution for ϕ_1.

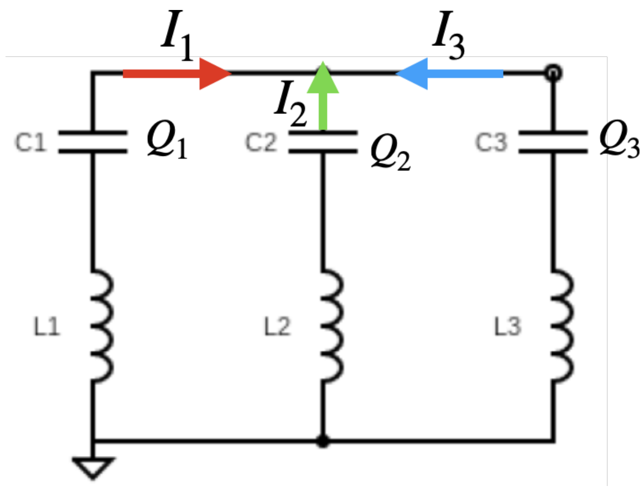
Total sub-points : 7

3. **Coupled LC oscillators:** Find expressions for the eigenfrequencies of the following electrical coupled circuit.



[Solution]

Consider the figure below



At the junction above $C2$, by Kirchhoff's rule, we have

$$I_1 + I_3 + I_2 = 0 \tag{90}$$
$$I_1 + I_2 = -I_3 \tag{91}$$

The charges and currents are related by

$$I_1 = \dot{Q}_1, \quad I_2 = \dot{Q}_2, \quad I_3 = \dot{Q}_3 \tag{92}$$

Recall that the energy stored in a capacitor (C) and inductor (L) is

$$\begin{cases} E_\text{C} = \frac{1}{2}\frac{Q^2}{C} \\ E_\text{L} = \frac{1}{2}LI^2 \end{cases} \tag{93}$$

respectively. Differentiating the above expressions with respect to time give power $P = VI$, and hence we have

$$\begin{cases} V_\text{C} I = \frac{Q}{C}\dot{Q} = \frac{Q}{C} I \quad \rightarrow \quad V_\text{C} = \frac{Q}{C} \\ V_\text{L} I = LI\dot{I} \quad \rightarrow \quad V_\text{L} = L\dot{I} \end{cases} \tag{94}$$

The voltage drop on the left, middle and right route are:

$$\begin{cases} V_\text{left} = \frac{Q_1}{C_1} + L_1\dot{I}_1 \\ V_\text{middle} = \frac{Q_2}{C_2} + L_2\dot{I}_2 \\ V_\text{right} = \frac{Q_3}{C_3} + L_3\dot{I}_3 \end{cases} \tag{95}$$

With $I_1 + I_2 = -I_3$, and since the routes are parallel, the voltage drop must be equal, giving

$$V_\text{left} = \frac{Q_1}{C_1} + L_1\dot{I}_1 = V_\text{right} = \frac{Q_3}{C_3} - L_3(\dot{I}_1 + \dot{I}_2) \tag{96}$$
$$(L_1 + L_3)\dot{I}_1 + L_3\dot{I}_2 = -\frac{Q_1}{C_1} + \frac{Q_3}{C_3} \tag{97}$$

and

$$V_{\text{middle}} = \frac{Q_2}{C_2} + L_2\dot{I}_2 = V_{\text{right}} = \frac{Q_3}{C_3} - L_3(\dot{I}_1 + \dot{I}_2) \tag{98}$$

$$L_3\dot{I}_1 + (L_2 + L_3)\dot{I}_2 = -\frac{Q_2}{C_2} + \frac{Q_3}{C_3} \tag{99}$$

Let $\frac{1}{C_1} = c_1$, $\frac{1}{C_2} = c_2$ and $\frac{1}{C_3} = c_3$, we have

$$\begin{cases} (L_1 + L_3)\dot{I}_1 + L_3\dot{I}_2 = -c_1Q_1 + c_3Q_3 \\ L_3\dot{I}_1 + (L_2 + L_3)\dot{I}_2 = -c_2Q_2 + c_3Q_3 \end{cases} \tag{100}$$

Differentiating 100 with respect to time, and with $I_1 = \dot{Q}_1$, $I_2 = \dot{Q}_2$ and $I_3 = \dot{Q}_3 = -(I_1 + I_2)$, we have

$$(L_1 + L_3)\ddot{I}_1 + L_3\ddot{I}_2 = -c_1I_1 - c_3(I_1 + I_2) \tag{101}$$

$$(L_1 + L_3)\ddot{I}_1 + L_3\ddot{I}_2 = -(c_1 + c_3)I_1 - c_3I_2 \tag{102}$$

and

$$L_3\ddot{I}_1 + (L_2 + L_3)\ddot{I}_2 = -c_2I_2 - c_3(I_1 + I_2) \tag{103}$$

$$L_3\ddot{I}_1 + (L_2 + L_3)\ddot{I}_2 = -c_3I_1 - (c_2 + c_3)I_2 \tag{104}$$

From the above two equations, we also have

$$L_1\ddot{I}_1 - L_2\ddot{I}_2 = -c_1I_1 + c_2I_2 \tag{105}$$

Equation (102)/$(L_1 + L_3)$ - Equation (105)/L_1 gives

$$\left(\frac{L_3}{L_1 + L_3} + \frac{L_2}{L_1}\right)\ddot{I}_2 = \left(-\frac{c_1 + c_3}{L_1 + L_3} + \frac{c_1}{L_1}\right)I_1 + \left(-\frac{c_3}{L_1 + L_3} - \frac{c_2}{L_1}\right)I_2 \tag{106}$$

$$\frac{L_1L_2 + L_1L_3 + L_2L_3}{L_1(L_1 + L_3)}\ddot{I}_2 = \frac{L_3c_1 - L_1c_3}{L_1(L_1 + L_3)}I_1 - \frac{L_1c_3 + L_1c_2 + L_3c_2}{L_1(L_1 + L_3)}I_2 \tag{107}$$

$$\ddot{I}_2 = \frac{L_3c_1 - L_1c_3}{L_1L_2 + L_1L_3 + L_2L_3}I_1 - \frac{L_1c_3 + L_1c_2 + L_3c_2}{L_1L_2 + L_1L_3 + L_2L_3}I_2 \tag{108}$$

Also, Equation (104)/$(L_2 + L_3)$ + Equation (105)/L_2 gives

$$\left(\frac{L_3}{L_2 + L_3} + \frac{L_1}{L_2}\right)\ddot{I}_1 = \left(-\frac{c_3}{L_2 + L_3} - \frac{c_1}{L_2}\right)I_1 + \left(-\frac{c_2 + c_3}{L_2 + L_3} + \frac{c_2}{L_2}\right)I_2 \tag{109}$$

$$\frac{L_1L_2 + L_1L_3 + L_2L_3}{L_2(L_2 + L_3)}\ddot{I}_1 = -\frac{L_2c_3 + L_2c_1 + L_3c_1}{L_2(L_2 + L_3)}I_1 + \frac{c_2L_3 - L_2c_3}{L_2(L_2 + L_3)}I_2 \tag{110}$$

$$\ddot{I}_1 = -\frac{L_2c_3 + L_2c_1 + L_3c_1}{L_1L_2 + L_1L_3 + L_2L_3}I_1 + \frac{c_2L_3 - L_2c_3}{L_1L_2 + L_1L_3 + L_2L_3}I_2 \tag{111}$$

Hence we can write the matrix equation as

$$\begin{bmatrix} \ddot{I}_1 \\ \ddot{I}_2 \end{bmatrix} = \begin{bmatrix} -\frac{L_2c_3 + L_2c_1 + L_3c_1}{L_1L_2 + L_1L_3 + L_2L_3} & \frac{c_2L_3 - L_2c_3}{L_1L_2 + L_1L_3 + L_2L_3} \\ \frac{L_3c_1 - L_1c_3}{L_1L_2 + L_1L_3 + L_2L_3} & -\frac{L_1c_3 + L_1c_2 + L_3c_2}{L_1L_2 + L_1L_3 + L_2L_3} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \tag{112}$$

For normal modes, $\ddot{I}_k = -\omega^2 I_k$, so we have

$$\begin{bmatrix} -\frac{L_2c_3+L_2c_1+L_3c_1}{L_1L_2+L_1L_3+L_2L_3}+\omega^2 & \frac{c_2L_3-L_2c_3}{L_1L_2+L_1L_3+L_2L_3} \\ \frac{L_3c_1-L_1c_3}{L_1L_2+L_1L_3+L_2L_3} & -\frac{L_1c_3+L_1c_2+L_3c_2}{L_1L_2+L_1L_3+L_2L_3}+\omega^2 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{113}$$

For non-trivial solutions, we require

$$\begin{vmatrix} -\frac{L_2c_3+L_2c_1+L_3c_1}{L_1L_2+L_1L_3+L_2L_3}+\omega^2 & \frac{c_2L_3-L_2c_3}{L_1L_2+L_1L_3+L_2L_3} \\ \frac{L_3c_1-L_1c_3}{L_1L_2+L_1L_3+L_2L_3} & -\frac{L_1c_3+L_1c_2+L_3c_2}{L_1L_2+L_1L_3+L_2L_3}+\omega^2 \end{vmatrix} = 0$$

$$\frac{(L_2c_3+L_2c_1+L_3c_1)(L_1c_3+L_1c_2+L_3c_2)}{(L_1L_2+L_1L_3+L_2L_3)^2} - \frac{L_2c_3+L_2c_1+L_3c_1+L_1c_3+L_1c_2+L_3c_2}{L_1L_2+L_1L_3+L_2L_3}\omega^2 + \omega^4 - \frac{(c_2L_3-L_2c_3)(L_3c_1-L_1c_3)}{(L_1L_2+L_1L_3+L_2L_3)^2} = 0$$

$$\frac{(L_1L_2+L_2L_3+L_1L_3)(c_1c_2+c_2c_3+c_1c_3)}{(L_1L_2+L_1L_3+L_2L_3)^2} - \frac{L_2c_3+L_2c_1+L_3c_1+L_1c_3+L_1c_2+L_3c_2}{L_1L_2+L_1L_3+L_2L_3}\omega^2 + \omega^4 = 0$$

$$\underbrace{(c_1c_2+c_2c_3+c_1c_3)}_{C}\underbrace{-(L_2c_3+L_2c_1+L_3c_1+L_1c_3+L_1c_2+L_3c_2)}_{B}\omega^2 + \underbrace{(L_1L_2+L_1L_3+L_2L_3)}_{A}\omega^4 = 0$$

Solving for ω^2, we have

$$\omega^2 = \frac{-B \pm \sqrt{B^2-4AC}}{2A} \tag{114}$$

The expression is too **horrible** to be written. The final expression inside the square root is

$$L_2^2c_3^2 + L_2^2c_1^2 + L_3^2c_1^2 + L_1^2c_3^2 + L_1^2c_2^2 + L_3^2c_2^2 \tag{115}$$

$$+2L_2^2c_1c_3 - 2L_2L_3c_1c_3 + 2L_1L_2c_3^2 - 2L_1L_2c_2c_3 - 2L_2L_3c_2c_3 + 2L_2L_3c_1^2 \tag{116}$$

$$-2L_1L_2c_1c_3 - 2L_1L_2c_1c_2 - 2L_2L_3c_1c_2 - 2L_1L_3c_1c_3 - 2L_1L_3c_1c_2 + 2L_3^2c_1c_2 \tag{117}$$

$$+2L_1^2c_2c_3 - 2L_1L_3c_2c_3 + 2L_1L_3c_2^2 \tag{118}$$

Supplementary box 4.3: Am I obtaining the right answer?
Looking at the horrible final expression, it seems that there is no way we can tell whether we are correct or not. We may do some simple consistency check.
If we set $L_1 = L_2 = L_3$ and $C_1 = C_2 = C_3$, the characteristic equation will reduce to

$$3c^2 - 6Lc\omega^2 + 3L^2\omega^4 = 0 \tag{119}$$

$$L^2\omega^4 - 2LC\omega^2 + c^2 = 0 \tag{120}$$

$$(L\omega^2 + c)^2 = 0\omega = \frac{c}{L} = \frac{1}{LC} \tag{121}$$

In this case, the system is said to be degenerate and hence the eigenfrequencies become the same - $\frac{1}{LC}$, which is the resonant frequency of a simple LC circuit. Indeed, this degenerated system is analogous to a block of mass $m = L$ connected to a spring of spring constant $k = C^{-1}$, in which case the resonant frequency is then $\omega^2 = k/m$. The charges Q in the circuit is then the displacement x of the block. If there is any resistance $R = \gamma L$ in the circuit, it is analogous to the damping force coefficient in mechanical problems γm.
Feynman Lecture Volume I Chapter 23 provides a detailed discussion on the similarity between mechanical systems and circuits. Take a look if time allows! =)

- **(6 pt)** : Formulating the decoupled matrix equation. If the student attempts to do so, but fails at the end, partial credit should still be given for the effort. Note that if the student uses the energy matrices (kinetic energy and potential energy matrices), credit should also be given.
- **(4 pt)** : Solving the characteristic equation.

Total sub-points : 10

4. **Triple coupled pendulums:** (H&F problem 3.2) A "triple pendulum" (three pendulums each coupled to the adjacent two via springs) has kinetic and potential energies given by

$$T = \frac{1}{2}\left(\dot{\theta_1}^2 + \dot{\theta_2}^2 + \dot{\theta_3}^2\right) \tag{122}$$

$$V = \frac{1}{2}(\theta_1^2 + \theta_2^2 + \theta_3^2 - 2\epsilon[\theta_1\theta_2 + \theta_2\theta_3 + \theta_3\theta_1]) \tag{123}$$

respectively, where m, g and l have been set equal to one for convenience, and ϵ contains the spring constant. (Don't try to picture this!)

(a) What are the T and V matrices?
[Solution]
The kinetic energy matrix is

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{124}$$

and the potential energy matrix is

$$\mathbf{V} = \begin{bmatrix} 1 & -\epsilon & -\epsilon \\ -\epsilon & 1 & -\epsilon \\ -\epsilon & -\epsilon & 1 \end{bmatrix} \tag{125}$$

- **(1 pt)** : Correct kinetic energy matrix T.
- **(1 pt)** : Correct potential energy matrix V.

Total sub-points : 2

(b) Find the normal mode frequencies.

[Solution]

To find the normal mode frequencies, we need to solve

$$|\mathbf{V} - \omega^2 \mathbf{T}| = 0 \tag{126}$$

$$\begin{vmatrix} 1-\omega^2 & -\epsilon & -\epsilon \\ -\epsilon & 1-\omega^2 & -\epsilon \\ -\epsilon & -\epsilon & 1-\omega^2 \end{vmatrix} = 0 \tag{127}$$

$$\begin{vmatrix} 1+\epsilon-\omega^2 & -(1+\epsilon-\omega^2) & 0 \\ 0 & 1+\epsilon-\omega^2 & -(1+\epsilon-\omega^2) \\ -\epsilon & -\epsilon & 1-\omega^2 \end{vmatrix} = 0 \tag{128}$$

$$(1+\epsilon-\omega^2)^2 \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -\epsilon & -\epsilon & 1-\omega^2 \end{vmatrix} = 0 \tag{129}$$

$$(1+\epsilon-\omega^2)^2 \left(\begin{vmatrix} 1 & -1 \\ -\epsilon & 1-\omega^2 \end{vmatrix} + \begin{vmatrix} 0 & -1 \\ -\epsilon & 1-\omega^2 \end{vmatrix} \right) = 0 \tag{130}$$

$$(1+\epsilon-\omega^2)^2(1-\omega^2-\epsilon-\epsilon) = 0 \tag{131}$$

$$(1+\epsilon-\omega^2)^2(1-2\epsilon-\omega^2) = 0 \tag{132}$$

where we have done “Row 1-Row2” and “Row 2 - Row3” in step 3, factored out $(1+\epsilon-\omega)^2$ in step 4, and done a cofactor expansion along the first row in step 5.

The normal mode frequencies are

$$\omega^2 = 1+\epsilon \quad \text{or} \quad 1-2\epsilon \tag{133}$$

- **(1 pt)** : Writing down the characteristic equation.
- **(1 pt)** : Solving the characteristic equation.
- **(2 pt)** : Correct normal mode frequencies. One point for each correct answer.

Total sub-points : 4

(c) Are there any restrictions on the value of the constant ϵ?

[Solution]

Since the normal mode frequencies cannot be complex (otherwise the solutions will become exponentials, which will die out eventually!), we have

$$\begin{cases} 1+\epsilon \geq 0 & \rightarrow \quad \epsilon \geq -1 \\ 1-2\epsilon \geq 0 & \rightarrow \quad \epsilon \leq \frac{1}{2} \end{cases} \tag{134}$$

Together, we get

$$-1 \leq \epsilon \leq \frac{1}{2} \tag{135}$$

- **(1 pt)** : Stating that the normal mode frequencies have to be real.
- **(2 pt)** : Writing down the constraint for ϵ.

Total sub-points : 3

(d) Show that a particular set of mode eigenvectors can be chosen to be

$$x_1 = \sqrt{\frac{2}{3}}(\theta_1, \theta_2, \theta_3), \tag{136}$$

$$x_2 = (\theta_1, 0, -\theta_3), \tag{137}$$

$$x_3 = \sqrt{\frac{1}{3}}(\theta_1, -2\theta_2, \theta_3) \tag{138}$$

and associate the frequencies from above with each mode.

[Solution]

For $\omega^2 = 1 + \epsilon$, we have the matrix equation as

$$\begin{cases} [1-(1+\epsilon)]\theta_1 - \epsilon\theta_2 - \epsilon\theta_3 = 0 \\ -\epsilon\theta_1 + [1-(1+\epsilon)]\theta_2 - \epsilon\theta_3 = 0 \\ -\epsilon\theta_1 - \epsilon\theta_2 + [1-(1+\epsilon)]\theta_3 = 0 \end{cases} \rightarrow \begin{cases} -\epsilon\theta_1 - \epsilon\theta_2 - \epsilon\theta_3 = 0 \\ -\epsilon\theta_1 - \epsilon\theta_2 - \epsilon\theta_3 = 0 \\ -\epsilon\theta_1 - \epsilon\theta_2 - \epsilon\theta_3 = 0 \end{cases} \tag{139}$$

which yields

$$\theta_1 + \theta_2 + \theta_3 = 0 \tag{140}$$

Note that the frequency $\omega = 1 + \epsilon$ is degenerated. Therefore two modes will be associated to this frequency.

If we choose $\theta_2 = 0$, then we have

$$\theta_3 = -\theta_1 \tag{141}$$

which corresponds to the eigenvector

$$x_2 = (\theta_1, 0, -\theta_3). \tag{142}$$

If we pick $\theta_1 = \theta_3$, then

$$\theta_2 = -2\theta_1 = -2\theta_3 \tag{143}$$

which corresponds to the eigenvector

$$x_3 = \sqrt{\frac{1}{3}}(\theta_1, -2\theta_2, \theta_3). \tag{144}$$

The coefficient in front is a constant related to the amplitude of the mode.

For $\omega^2 = 1 - 2\epsilon$, we have the matrix equation as

$$\begin{cases} [1-(1-2\epsilon)]\theta_1 - \epsilon\theta_2 - \epsilon\theta_3 = 0 \\ -\epsilon\theta_1 + [1-(1-2\epsilon)]\theta_2 - \epsilon\theta_3 = 0 \\ -\epsilon\theta_1 - \epsilon\theta_2 + [1-(1-2\epsilon)]\theta_3 = 0 \end{cases} \rightarrow \begin{cases} 2\epsilon\theta_1 - \epsilon\theta_2 - \epsilon\theta_3 = 0 \\ -\epsilon\theta_1 + 2\epsilon\theta_2 - \epsilon\theta_3 = 0 \\ -\epsilon\theta_1 - \epsilon\theta_2 + 2\epsilon\theta_3 = 0 \end{cases} \tag{145}$$

which yields

$$\begin{cases} 3\epsilon\theta_1 - 3\epsilon\theta_2 = 0 \\ 3\epsilon\theta_2 - 3\epsilon\theta_3 = 0 \end{cases} \quad \rightarrow \quad \begin{cases} \theta_1 = \theta_2 \\ \theta_2 = \theta_3 \end{cases} \tag{146}$$

and hence

$$\theta_1 = \theta_2 = \theta_3 \tag{147}$$

which corresponds to the eigenvector

$$x_1 = \sqrt{\frac{2}{3}}(\theta_1, \theta_2, \theta_3) \tag{148}$$

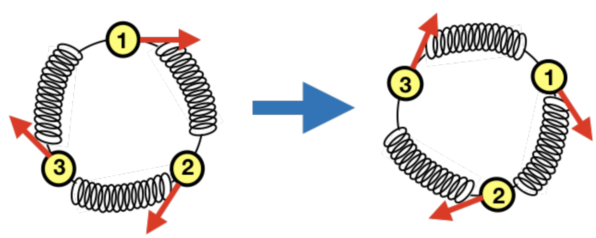
- **(1 pt)** : Solving for the eigenvectors for $\omega = 1 + \epsilon$.
- **(4 pt)** : Showing that x_2 and x_3 are eigenvectors associated to $\omega = 1 + \epsilon$.
- **(1 pt)** : Solving for the eigenvectors for $\omega = 1 - 2\epsilon$.
- **(2 pt)** : Showing that x_1 is an eigenvector associated to $\omega = 1 - 2\epsilon$.

Total sub-points : 8

(e) Draw these modes.

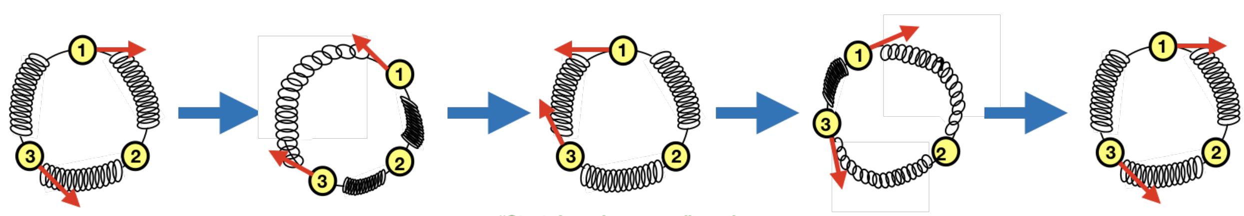
[Solution]

For the normal mode corresponding to x_1:



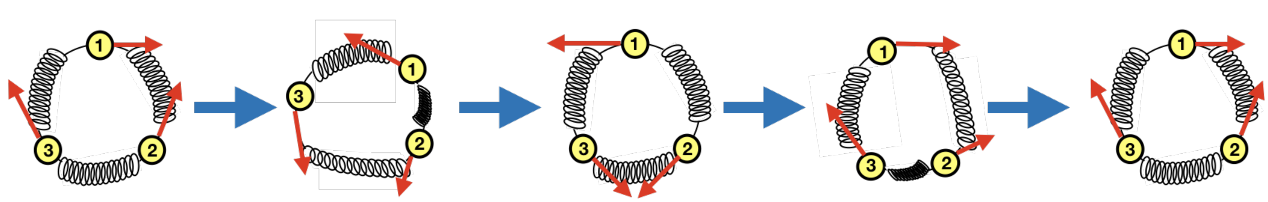
Pure rotation mode

For the normal mode corresponding to x_2:



“Stretch and squeeze” mode

For the normal mode corresponding to x_3:



“Stretch and squeeze” mode

- **(2 pt)** : Sketching a complete cycle for normal mode x_1. Note that if students instead draw the eigenvector in the $\theta_1 - \theta_2 - \theta_3$ $3D$ space, credit should also be given. Same rubric applies to the following two items.
- **(2 pt)** : Sketching a complete cycle for normal mode x_2.
- **(2 pt)** : Sketching a complete cycle for normal mode x_3.

Total sub-points : 6

Maximum points obtainable for this problem set: 77 pt