Physics 106a: Problem Set 3 – Solutions

October 19, 2019

Problem 1: Holonomic and nonholonomic

When we denote the center of one wheel on the (x, y)-plane as (x_1, y_1) and the center of the other wheel as (x_2, y_2), they satisfy

$$\dot{x}_i = a\dot{\phi}_i \sin\theta, \tag{1}$$
$$\dot{y}_i = -a\dot{\phi}_i \cos\theta, \tag{2}$$

because each wheel rolls without slipping. Using $x = (x_1 + x_2)/2$ and $y = (y_1 + y_2)/2$, we get two nonholonomic constraints

$$\cos\theta\delta x + \sin\theta\delta y = 0, \tag{3}$$
$$\sin\theta\delta x - \cos\theta\delta y = \frac{1}{2}a(\delta\phi_1 + \delta\phi_2). \tag{4}$$

Another constraint is obtained by considering the rotation of the axle,

$$\dot{x}_1 - \dot{x}_2 = b\dot{\theta}\sin\theta, \tag{5}$$
$$\dot{y}_1 - \dot{y}_2 = -b\dot{\theta}\cos\theta. \tag{6}$$

Using the relation between $\dot{x}_i$ (or $\dot{y}_i$) and $\dot{\phi}_i$, we get

$$a(\dot{\phi}_1 - \dot{\phi}_2) = b\dot{\theta}. \tag{7}$$

After integrating this relation with respect to time, we get a holonomic constraint

$$b\theta = a(\phi_1 - \phi_2) + \text{constant}. \tag{8}$$

Problem 2: Particle constrained to elliptical wire (revisited)

(a) The kinetic energy of the system is $T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right)$, and the potential energy is mgy. So, the Lagrangian is $L = T - V = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mgy$.

(b) The constraint of the system is $G(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$. Using the Lagrange multiplier method, we can introduce a new effective Lagrangian $L' = L + \lambda G = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mgy + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$.

$$\frac{\partial L'}{\partial x} = \lambda \frac{2x}{a^2} \tag{9}$$

$$\frac{\partial L'}{\partial \dot{x}} = m\dot{x} \tag{10}$$

$$\frac{\partial L'}{\partial y} = -mg + \lambda \frac{2y}{b^2} \tag{11}$$

$$\frac{\partial L'}{\partial \dot{y}} = m\dot{y} \tag{12}$$

Using Hamilton's principle, the EOM is

$$\begin{cases} m\ddot{x} = \dfrac{2\lambda x}{a^2} \\ m\ddot{y} = -mg + \dfrac{2\lambda y}{b^2} \end{cases} \tag{13}$$

Cancel out λ in both equation

$$\Rightarrow \frac{a^2\ddot{x}}{x} = \frac{b^2\left(\ddot{y} + g\right)}{y} \tag{14}$$

(c) Let $x = a\cos\alpha$, $y = b\sin\alpha$ in equation (14), we can get

$$\frac{a^2\left(-a\cos\alpha\dot{\alpha}^2 - a\sin\alpha\ddot{\alpha}\right)}{a\cos\alpha} = \frac{b^2\left(-b\sin\alpha\dot{\alpha}^2 + b\cos\alpha\ddot{\alpha} + g\right)}{b\sin\alpha} \tag{15}$$

$$\Rightarrow \left(a^2\sin^2\alpha + b^2\cos^2\alpha\right)\ddot{\alpha} + \left(a^2 - b^2\right)\sin\alpha\cos\alpha\dot{\alpha}^2 + gb\cos\alpha = 0 \tag{16}$$

This is exactly the same answer derived directly using the generalized coordinate α in assignment 1.

(d) From equation (13), it is easy to get $\lambda = \frac{ma^2\ddot{x}}{2x} = -\frac{1}{2}ma^2\left(\dot{\alpha}^2 + \tan\alpha\ddot{\alpha}\right)$.
Furthermore, comparing equation (13) with the EOM derived from Newtonian mechanics ($m\ddot{x} = \mathcal{N}_x$ and $m\ddot{y} = -mg + \mathcal{N}_y$), we have

$$\mathcal{N}_x = \frac{2\lambda x}{a^2} \quad \text{and} \quad \mathcal{N}_y = \frac{2\lambda y}{b^2}. \tag{17}$$

This is consistent with the expected result relating λ to the constraint force

$$\mathcal{N} = \lambda\boldsymbol{\nabla}G\ . \tag{18}$$

To check that the constraint force is perpendicular to the wire we calculate the tangent vector first. The constraint $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ implies that the tangent slope satisfies $\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$ so that the tangent vector $\mathbf{t}$ is proportional to $(y/b^2, -x/a^2)$. You can now see $\mathcal{N} \cdot \mathbf{t} = 0$.

Problem 3: Bead on a spinning hoop

(a) The constraint is *holonomic* and *rheonomic* (taking the origin at the center of the hoop the position vector in Cartesian coordinates $\vec{r} = (R\cos(\Omega t)\sin\theta, R\sin(\Omega t)\sin\theta, R\cos\theta)$, which gives *time dependent* constraint equations.

(b) The generalized force can be found by writing down the gravitational force in Cartesian coordinates,

$$\vec{F} = -mg\hat{z} \tag{19}$$

and from the expression for the vector $\vec{r}$,

$$\mathcal{F}_\theta = \vec{F} \cdot \frac{\partial \vec{r}}{\partial \theta} = -mgR\sin\theta. \tag{20}$$

You should not add terms from fictitious (inertial) forces: such effects appear naturally from the generalized equation of motion or the Euler-Lagrange equation, and you would be double counting them.

(c) Using the generalized coordinate θ, the kinetic energy can be written as,

$$T = \frac{1}{2}m\frac{d\vec{r}}{dt}^2 = \frac{1}{2}mR^2\dot{\theta}^2 + \frac{1}{2}mR^2\sin\theta^2\Omega^2. \tag{21}$$

The potential energy is $V = -mgR\cos\theta$.

(d) The equation of motion is given by,

$$\begin{aligned} mR^2\ddot{\theta} &= mR^2\sin\theta\cos\theta\Omega^2 - mgR\sin\theta \\ \rightarrow \ddot{\theta} &+ \sin\theta\left(-\Omega^2\cos\theta + \frac{g}{R}\right) = 0. \end{aligned} \tag{22}$$

(e) The Hamiltonian is,

$$\mathcal{H} = \frac{\partial L}{\partial \dot{\theta}}\dot{\theta} - L = \frac{1}{2}mR^2\dot{\theta}^2 - \frac{1}{2}mR^2\Omega^2\sin^2\theta - mgR\cos\theta. \tag{23}$$

The Hamiltonian is a constant of the motion since there is no explicit time dependence in the Lagrangian.

(f) The total energy $T + V$, not equal to the Hamiltonian in this case, is *not* a constant of the motion, although the Hamiltonian is conserved. The energy is not conserved because the constraint force may do work in the *actual* displacement of the bead in the dynamics, even though it does no work in a *virtual* displacement, and so we must do more work keeping the wire rotating at a fixed rate.

(g) The condition for an equilibrium point is $\partial L/\partial\theta = 0$, which gives

$$mR^2\Omega^2\sin\theta\cos\theta - mgR\sin\theta = 0 \tag{24}$$

Since $\sin\theta = 0$ when $\theta = \{0, \pi\}$, these are equilibria for all rotation rates.

To determine the stability of each, simply expand the equation of motion in small angles. For $\theta = 0$,

$$\ddot{\delta\theta} = \left(\Omega^2 - \frac{g}{R}\right)\delta\theta, \tag{25}$$

and for $\theta = \pi$,

$$\ddot{\delta\theta} = \left(\Omega^2 + \frac{g}{R}\right) \delta\theta. \tag{26}$$

We see that the stability of $\theta = 0$ depends critically on the angular frequency: if $1 < g/(R\Omega^2)$ then $\theta = 0$ is stable, and if $1 > g/(R\Omega^2)$ then $\theta = 0$ is unstable. The equilibrium point $\theta = \pi$ is always unstable.

(h) From (24) if $\Omega^2 > \Omega_c^2 = g/R$ then two more equilibrium points appear at $\pm\theta_r = \arccos(g/(\Omega^2 R))$.

(i) The behavior near the new equilibria can be found from the effective potential from Eq. (23)

$$V_{\text{eff}}(\theta) = -\tfrac{1}{2} m R^2 \Omega^2 \sin^2\theta - mgR\cos\theta \quad \Rightarrow \quad V''_{\text{eff}}(\pm\theta_r) = -mR^2\Omega^2[1 - (\Omega_c/\Omega)^4]. \tag{27}$$

The equilibria $\pm\theta_r$ are therefore always stable when they exist, and the frequency of oscillation about them is

$$\omega^2 = \Omega^2 \left(1 - \frac{\Omega_c^4}{\Omega^4}\right). \tag{28}$$

(j) The Lagrangian in Cartesian coordinates is

$$L = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + mgz + \text{constant}. \tag{29}$$

There are two constraints to confine the mass to a curve in the three dimensional space. For example

$$x^2 + y^2 + z^2 - R^2 = 0, \tag{30}$$
$$y - x\tan\Omega t = 0. \tag{31}$$

The effective action is

$$\bar{S} = \int dt \left[L + \lambda_1(t)(x^2 + y^2 + z^2 - R^2) + \lambda_2(t)(y - x\tan\Omega t)\right]. \tag{32}$$

Other choices are possible. Note that you should not put velocity constraints into the effective action.

Problem 4: Particle in a bowl: (Hand and Finch 3.1)

A point particle of mass m is confined to the frictionless surface of a half-spherical bowl. In the presence of a uniform gravitational acceleration g. There are two degrees of freedom.

(a) Prove that the equilibrium point is the bottom of the bowl.

[Solution]

From the lecture notes, we know that the condition for an equilibrium is

$$\left.\frac{\partial \mathcal{L}}{\partial q_k}\right|_{\dot{q}_k=0} = 0, \tag{33}$$

for Lagrangians $\mathcal{L}$ with no explicit time dependence. In particular, if the constraints are *holonomic*, then the kinetic energy term T is a quadratic form in $\{\dot{q}_k\}$ (You have proved this in question 3 of problem set 2) and so at equilibrium $\frac{\partial T}{\partial q_k} = 0$ since $\dot{q}_k = 0$. In this case, the equilibrium condition can be simplified as

$$\frac{\partial V}{\partial q_k} = 0. \tag{34}$$

Now, in this problem the mass is constrained to move on a half-spherical bowl. Using r, θ and ϕ as the generalized coordinates (where the origin is set to be at the center of the half-spherical bowl, and $\theta = 0$ when the mass is vertically below the origin), the Lagrangian is simply

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2(\theta)\dot{\phi}^2) + mgr\cos\theta \tag{35}$$

subjected to the constraint $f = R - r = 0$, where R, the radius of the bowl, is a constant. With such we can simplify the Lagrangian as

$$\mathcal{L} = \frac{1}{2}mR^2(\dot{\theta}^2 + \sin^2(\theta)\dot{\phi}^2) + mgR\cos\theta \tag{36}$$

Note that the constraint is *holonomic*. Therefore, for the equilibrium position we require

$$\begin{aligned} \frac{\partial}{\partial \theta}(mgR\cos\theta) &= 0 \\ -mgR\sin\theta &= 0 \\ \theta = 0 \text{ or } \pi(\text{rejected*}) \end{aligned}$$

which proves that the equilibrium point is the bottom of the bowl (*Note: $\theta = \pi$ refers to some other position, see **Supplementary box 3.1**.)

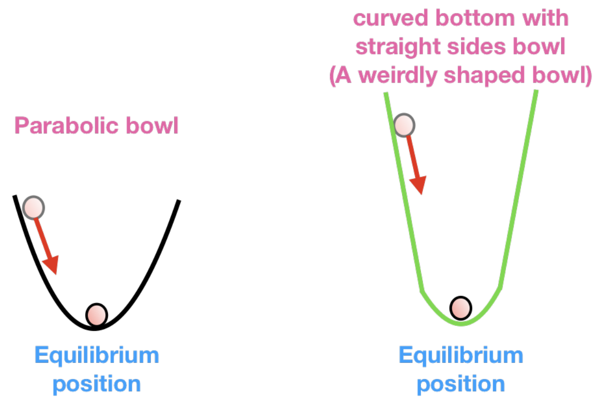
- **(1 pt)** : Realising that the constraint to this problem is holonomic (Or, writing down the Lagrangian of this problem with correct constraint).
- **(1 pt)** : Applying the equilibrium condition on either the Lagrangian, or the potential energy function.
- **(2 pt)** : Solving for θ and show that $\theta = 0$ for the equilibrium point.

Total sub-points : 4

(b) Does the bowl have to be exactly spherical for this to be true? Near to the bottom of the bowl, what is the most general form possible for the shape of the bowl in order to maintain the stability of the equilibrium point at the bottom?

[Solution]

By daily observations you should know that the bowl DOES NOT have to be exactly spherical for the result in part (a) to be true.



However, there are special requirements regarding the shape *near the bottom* of the bowl.

Suppose the shape of the bowl can be described as $z = f(r, \theta)$ in terms of *cylindrical coordinates*, and $f(r, \theta)$ is some general function in terms of r and θ. The potential energy function is then (in general)

$$V = -f(r, \theta)mg.$$

We require the bottom of the bowl ($r = r_0$ =Distance from the origin to the bottom of the bowl, $\theta = 0$) to be the equilibrium position. This means that the bottom of the bowl has to be a local minimum point for the potential function (or, the potential function has to be a constant around the bottom of the bowl) We expand the potential energy function about the bottom of the bowl (note that V is independent of ϕ, and so we excluded the terms related to it):

$$V(r,\theta) \approx V(r_0,0) + \left[\frac{\partial V}{\partial r}\bigg|_{r_0,0}(r-r_0) + \frac{\partial V}{\partial \theta}\bigg|_{r_0,0}\theta\right] + \frac{1}{2}\left[\frac{\partial^2 V}{\partial r^2}\bigg|_{r_0,0}(r-r_0)^2 + 2\frac{\partial^2 V}{\partial r\partial\theta}\bigg|_{r_0,0}(r-r_0)\theta + \frac{\partial^2 V}{\partial \theta^2}\bigg|_{r_0,0}\theta^2\right]$$

$$V(r,\theta) \approx -mgV_{\min} - mg\left[\frac{\partial f}{\partial r}\bigg|_{r_0,0}(r-r_0) + \frac{\partial f}{\partial \theta}\bigg|_{r_0,0}\theta\right] + \frac{1}{2}\left[\frac{\partial^2 f}{\partial r^2}\bigg|_{r_0,0}(r-r_0)^2 + 2\frac{\partial^2 f}{\partial r\partial\theta}\bigg|_{r_0,0}(r-r_0)\theta + \frac{\partial^2 f}{\partial \theta^2}\bigg|_{r_0,0}\theta^2\right]$$

but we know that for the bottom to be an equilibrium point, the 1$^{\text{st}}$ derivative term must vanish, and the 2nd derivative must be larger than zero (concave upward)! This means that we must have

$$\begin{cases} \frac{\partial f}{\partial r}\big|_{r_0,0}(r-r_0) + \frac{\partial f}{\partial \theta}\big|_{r_0,0}\theta = 0, \text{ and} \\ \frac{\partial^2 f}{\partial r^2}\big|_{r_0,0}(r-r_0)^2 + 2\frac{\partial^2 f}{\partial r\partial\theta}\big|_{r_0,0}(r-r_0)\theta + \frac{\partial^2 f}{\partial \theta^2}\big|_{r_0,0}\theta^2 > 0 \end{cases} \tag{37}$$

Hence, the most general form possible for the shape of the bowl near to the bottom must satisfy the above two equation. Possible shapes that fulfil the above conditions are *spheres*, *ellipsoid* and *paraboloids*.

Supplementary box 3.1: Types of equilibria, and beyond

From your answer in part (a), you found that $\theta = \pi$ can also fulfil the equilibrium condition. Which position does this solution refer to?

In fact, the $\theta = \pi$ can also refer to a position also at the bottom of a half spherical bowl, but this time the bowl is *inverted*. In either of the cases, the bottom of the bowl is also a equilibrium point. The only difference is that one of them is *stable*, and one of them is *unstable*.

To see whether an equilibrium point is stable, we expand the potential function about the equilibrium point up to the second order term, and solve the Lagrangian with the approximated potential function. If we keep only the second order terms, we will be ending up with an equation in the form

$$\ddot{q} + Aq = 0, \tag{38}$$

where q is a generalized coordinate, and A is some constant.

We see that if A is smaller than 0, the solution for q will be in the form

$$q = C_1 e^{\sqrt{A}t} + C_2 e^{-\sqrt{A}t}, \tag{39}$$

in which the term $C_1 e^{\sqrt{A}t}$ can go to very large values as time t. This means that the mass will not return to the equilibrium position under a small perturbation. The corresponding equilibrium point is hence called *unstable*.

On the other hand, if A is larger than 0, the solution for q will be in the form

$$q = C_1 \sin(\sqrt{A}t) + C_2 \cos(\sqrt{A}t), \tag{40}$$

in which both terms will only oscillate around the equilibrium position. This means that the mass will always return to the equilibrium position even under a small perturbation. The corresponding equilibrium point is hence called *stable*.

Bonus question 1:
Redo part (b) with the potential in (a) around the equilibrium point $r = r_0, \theta = 0$ ($r = r_0, \theta = \pi$) and show that the equilibrium is stable (unstable).

Readers should also question if the solution in (b) does not include a flat-bottom bowl. In fact, every point on the flat bottom is a *neutral* equilibrium point. A particle can be at equilibrium at any point on the flat bottom, but it is neither stable nor unstable, since under a perturbation the particle will move to another point on the bottom and return to equilibrium state. It is not necessary for the particle to return to its original position.

Bonus question 2:
Show that a flat-bottom bowl fulfils the equilibrium conditions you found in part (b).

Bonus question 3:
We said that apart from a spherical bowl, an ellipsoidal bowl and a paraboloid bowl can also fulfil the equilibrium position.

(i) **Redo part (b) in terms of Cartesian coordinates.**

(ii) **The general ellipsoidal equation (with the bottom of the ellipsoid touching the origin) is given by**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{(z-c)^2}{c^2} = 1 \tag{41}$$

Show that the bottom point of general ellipsoid ($(x, y, z) = (0, 0, 0)$) satisfies the equilibrium conditions.

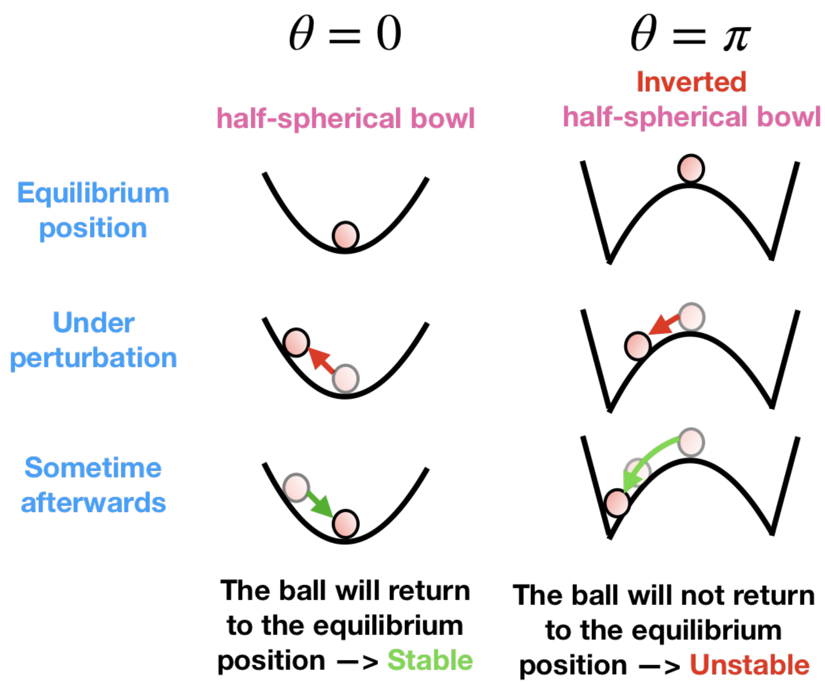
(iii) **The general equation for a paraboloid (with the bottom point at the origin) is given by**

$$z = k(x^2 + y^2) \tag{42}$$

Show that the bottom point of the paraboloid satisfy the equilibrium conditions.

What about an inverted cone shaped bowl? Is the bottom still an equilibrium position? (Hint: Can the particle ever reach the "real" bottom of the cone-shaped bowl? A more subtle point: Is the potential function differentiable at the bottom of the bowl?)

Finally, we have only been talking about a "point" particle in a bowl. What if the particle is a spherical object with a finite size (say with a radius b)? Is it still true that as long as the shape near the bottom is spherical, the bottom will remain an equilibrium point? Are there any other more constraints on the general shape of the bowl? Think about it.



- **(1 pt)** : Stating that the bowl does not have to be exactly spherical for the results in (a) to be true.
- **(1 pt)** : Writing down the potential energy function V in generalized coordinates. Note that Cartesian coordinates (x, y, z), cylindrical coordinates (r, θ, z) and spherical coordinates (r, θ, ϕ) are all acceptable answers.
- **(2 pt)** : Expanding V about the bottom of the bowl up to the second order term.
- **(2 pt)** : Showing that for V to have a local minimum at the bottom, one requires the first derivative of the shape function to vanish, and stating explicitly the mathematical expression for this.
- **(2 pt)** : Showing that for V to have a local minimum at the bottom, one requires the second derivative of the shape function to be larger than 0, and stating explicitly the mathematical expression for this.

Total sub-points : 8