Physics 106a: Problem Set 2 – Solutions

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Problem 1: Hanging spring

(a) In polar coordinates (r, θ), the mass m has velocity $\mathbf{v} = (\dot{r}, r\dot{\theta})$. Thus

$$\begin{aligned} T &= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)\,, \\ V &= -mgr\cos\theta + \frac{1}{2}k(r - l_0)^2\,. \end{aligned} \tag{1}$$

The Lagrangian of m is therefore

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + mgr\cos\theta - \frac{1}{2}k(r - l_0)^2\,. \tag{2}$$

(b)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \tag{3}$$

gives

$$m\ddot{r} - mr\dot{\theta}^2 - mg\cos\theta + k(r - l_0) = 0\,. \tag{4}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \tag{5}$$

gives

$$mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} + mgr\sin\theta = 0\,. \tag{6}$$

The rest length of the spring with mass m hanging, r_0, is given by Hooke's law

$$k(r_0 - l_0) = mg\,. \tag{7}$$

Thus with $\lambda = (r - r_0)/r_0$ we have

$$\begin{aligned} r - l_0 &= \lambda r_0 + \frac{mg}{k}\,, \\ r &= r_0(1 + \lambda)\,, \\ \dot{r} &= r_0\dot{\lambda}\,, \\ \ddot{r} &= r_0\ddot{\lambda}\,. \end{aligned} \tag{8}$$

and the equations of motion become

$$\begin{aligned} \ddot{\lambda} + \frac{k\lambda}{m} - (1 + \lambda)\dot{\theta}^2 + \frac{g}{r_0}(1 - \cos\theta) &= 0\,, \\ (1 + \lambda)\ddot{\theta} + 2\dot{\lambda}\dot{\theta} + \frac{g}{r_0}\sin\theta &= 0\,; \end{aligned} \tag{9}$$

or, with $\omega_s^2 = k/m$, $\omega_p^2 = g/r_0$,

$$\begin{aligned} \ddot{\lambda} + (\omega_s^2 - \dot{\theta}^2)\lambda - \dot{\theta}^2 + \omega_p^2(1 - \cos\theta) &= 0\,, \\ (1 + \lambda)\ddot{\theta} + 2\dot{\lambda}\dot{\theta} + \omega_p^2\sin\theta &= 0\,. \end{aligned} \tag{10}$$

(c) When λ and θ are small, we can neglect second order quantities in θ, λ, $\dot{\theta}$, $\dot{\lambda}$, and the equations of motion reduce to

$$\begin{aligned} \ddot{\lambda} + \omega_s^2 \lambda &= 0\,, \\ \ddot{\theta} + \omega_p^2 \theta &= 0 \end{aligned} \tag{11}$$

in the lowest order approximation. For the given initial conditions, we find

$$\begin{aligned} \lambda &= A\cos(\omega_s t)\,, \\ \theta &= B\sin(\omega_p t)\,. \end{aligned} \tag{12}$$

Thus λ and θ each oscillates sinusoidally with angular frequencies ω_s and ω_p respectively. They differ in phase by $\pi/2$.

(d) If we retain also terms of the second order, the equation of motion for λ becomes

$$\ddot{\lambda} + \omega_s^2 \lambda = \dot{\theta}^2 - \frac{1}{2}\omega_p^2 \theta^2\,. \tag{13}$$

Using the results of the lowest order approximation for θ, the equation can be approximated as

$$\ddot{\lambda} + \omega_s^2 \lambda \approx \frac{1}{2} B^2 \omega_p^2 [2\cos^2(\omega_p t) - \sin^2(\omega_p t)] = \frac{1}{4} B^2 \omega_p^2 [3\cos(2\omega_p t) + 1]\,. \tag{14}$$

Thus λ may resonate if $\omega_s = 2\omega_p$. However this is unlikely to realize physically since as the amplitude of λ increases toward a resonance the lowest order approximation no longer holds and higher order effects will take place. Furthermore the nonlinear properties of the spring will also come into play, invalidating the original simplified model.

Problem 2: Massive rope and pulley, frictional forces

(a) The forces acting are gravity, the frictional force from the oil, tension, and the normal force from the pulley. For the virtual displacement corresponding to a change δx the virtual work from the tension and the normal forces is zero. The gravity forces $-\rho g \delta l \hat{x}$ on an element of length δl give contributions to the virtual work that cancel except for the excess length $2x$ on the long side. The frictional force is $-\gamma \dot{x} \delta l$ parallel to the displacement for each δl (remember that velocities are held fixed in a the virtual displacement). Thus the virtual work is

$$\delta W = (2\rho g x - \gamma l \dot{x})\delta x. \tag{15}$$

(b) The generalized force is $\delta W/\delta x$ and is

$$\mathcal{F}_x = 2\rho g x - \gamma l \dot{x}. \tag{16}$$

(c) For D'Alembert's principle we need $\sum_{\text{elements}} \dot{\vec{p}} \cdot \delta\vec{r}$ with $\delta\vec{r}$ the virtual displacement. Since the rate of change of momentum of each element is parallel to the virtual displacement this is just $\rho l \ddot{x}\, \delta x$. Thus the equation of motion is

$$\rho l \ddot{x} = 2\rho g x - \gamma l \dot{x}. \tag{17}$$

The same equation is obtained from the Golden Rule #1, since $\partial T/\partial \dot{x} = m\dot{x}$ and $\partial T/\partial x = 0$. The equation of motion is of the given form with $\alpha = \gamma/\rho$ and $\beta = -2g/l$.

(d) The potential energy is that of the excess mass on one side $2\rho x$ at the position of its center of mass x below the initial end positions, i.e. $V = -\rho g x^2$. The Lagrangian is $L = \frac{1}{2}\rho l \dot{x}^2 + \rho g x^2$, and the equation of motion follows.

For $\gamma = 0$ the equation of motion is $\ddot{x} + \beta x = 0$. Since β is negative, the solutions are exponentials $x(t) \propto e^{\pm\sqrt{|\beta|}t}$. The initial conditions are satisfied by the combination

$$x(t) = x_0 \cosh\left(\sqrt{|\beta|}t\right) \quad \text{with} \quad \beta = -2g/l. \tag{18}$$

At the moment that the rope lose *full* contact with the pulley (there is still a part with length πR in contact), we have the displacement:

$$x' = l - \pi R - \frac{l - \pi R}{2} = \frac{l - \pi R}{2}. \tag{19}$$

By solving the equation of $x(t) = x'$, we have:

$$t' = \frac{1}{\sqrt{|\beta|}} \cosh^{-1}\left(\frac{l - \pi R}{2x_0}\right). \tag{20}$$

This is the time that the rope is in full contact with the pulley.

Note that virtual work provides the appropriate formalism for treating this problem. You might have attempted it in Ph1a, when there was no good way of talking about the acceleration of the masses "along the string" given by the net force.

We should also note that the model here is fairly idealized: we did not consider the horizontal motion of the rope, or we actually lost a lot of degree of freedom. One example is that the center-of-mass of the rope moves right-ward when it moves, which is not consistent with the conversation of momentum in the horizontal direction; however the model is still a good approximation as we assumed $l \gg \pi R$. Full solution to the motion should infer more assumptions like the properties of the rope material, which is outside the scope of the course.

Problem 3: The Hamiltonian - Quadratic forms

(a) Since the constraints are scleronomic (independent of time), we may express the Cartesian coordinate for the particle labeled i as

$$\vec{r}_i = \vec{r}_i(q_1, \cdots, q_N \cancel{, t}), \tag{21}$$

which does not contain t explicitly. Note that we assumed N generalized independent coordinates. We may evaluate the generalized velocity:

$$\vec{v}_i = \sum_{l=1}^{N} \frac{\partial \vec{r}_i}{\partial q_l} \frac{\mathrm{d}q_l}{\mathrm{d}t}. \tag{22}$$

As a result, we may evaluate the kinetic energy by summing over all M particles:

$$\begin{aligned} T =& \frac{1}{2} \sum_{i=1}^{M} m_i |\vec{v}_i|^2 \\ =& \frac{1}{2} \sum_{i=1}^{M} m_i \sum_{l=1}^{N} \frac{\partial \vec{r}_i}{\partial q_l} \dot{q}_l \cdot \sum_{m=1}^{N} \frac{\partial \vec{r}_i}{\partial q_m} \dot{q}_m \\ =& \frac{1}{2} \sum_{i=1}^{M} m_i \sum_{l,m}^{N} \frac{\partial \vec{r}_i}{\partial q_l} \cdot \frac{\partial \vec{r}_i}{\partial q_m} \dot{q}_l \dot{q}_m, \end{aligned} \tag{23}$$

which is a quadratic function of the generalized velocities. We may also find that

$$\begin{aligned} \sum_k \dot{q}_k \frac{\partial T}{\partial \dot{q}_k} =& \sum_k \dot{q}_k \cdot \frac{1}{2} \sum_{i=1}^{M} m_i \sum_{l,m}^{N} \frac{\partial \vec{r}_i}{\partial q_l} \cdot \frac{\partial \vec{r}_i}{\partial q_m} (\delta_{kl} \dot{q}_m + \dot{q}_l \delta_{km}) \\ =& \frac{1}{2} \sum_{i=1}^{M} m_i \sum_{k,m}^{N} \frac{\partial \vec{r}_i}{\partial q_k} \cdot \frac{\partial \vec{r}_i}{\partial q_m} \dot{q}_k \dot{q}_m + \frac{1}{2} \sum_{i=1}^{M} m_i \sum_{l,k}^{N} \frac{\partial \vec{r}_i}{\partial q_l} \cdot \frac{\partial \vec{r}_i}{\partial q_k} \dot{q}_l \dot{q}_k \\ =& 2T. \end{aligned} \tag{24}$$

(b) Since the potential energy V is not dependent on the generalized velocity $\dot{q}_k$, we have

$$\frac{\partial L}{\partial \dot{q}_k} = \frac{\partial T}{\partial \dot{q}_k}. \tag{25}$$

By definition and refer equation (24) we may find

$$\begin{aligned} H =& \sum_k \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L \\ =& 2T - (T - V) \\ =& T + V \\ =& E. \end{aligned} \tag{26}$$

For systems with scleronomic constraints, the Hamiltonian is the total energy.

Problem 4: Two ways to handle constraints

(a) Let's write the position of the mass as $\vec{r} = (x, \sin x)$. Then, $\dot{\vec{r}} = (\dot{x}, \dot{x}\cos x)$. The Lagrangian is given by

$$L = \tfrac{1}{2}m(\dot{x}^2 + \dot{x}^2\cos^2 x) - mg\sin x. \tag{27}$$

Now, we plug this into Euler-Lagrange equation to get

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = m(1+\cos^2 x)\ddot{x} - 2m\cos x\sin x(\dot{x})^2 + m\cos x\sin x(\dot{x})^2 + mg\cos x = 0 \tag{28}$$

Therefore, the equation of motion is

$$(1+\cos^2 x)\ddot{x} - \cos x\sin x(\dot{x})^2 + g\cos x = 0 \tag{29}$$

(b) We can treat x, z to be two independent dynamical variables and supplement the constraint on them by introducing the Lagrange multiplier. The Lagrangian can be written as

$$L' = \frac{1}{2}m(\dot{x}^2 + \dot{z}^2) - mgz - \lambda(z - \sin x). \tag{30}$$

Now, plugging into Euler-Lagrange equation for x and z, we get

$$m\ddot{x} - \lambda\cos x = 0 \tag{31}$$

$$m\ddot{z} + mg + \lambda = 0. \tag{32}$$

(c) Now using the constraint $z = \sin x$, (32) can be written as

$$\lambda = -mg - m(-\sin x(\dot{x})^2 + \ddot{x}\cos x). \tag{33}$$

Plugging the above equation to (31), we get

$$m\ddot{x} + \cos x(mg + m(-\sin x(\dot{x})^2 + \ddot{x}\cos x)) = 0 \tag{34}$$

$$(1+\cos^2 x)\ddot{x} - \cos x\sin x(\dot{x})^2 + g\cos x = 0 \tag{35}$$

which is the same result as (a).