Physics 106a: Assignment 1 (Solutions prepared by Alvin Li)

4 October, 2019; due 7pm Friday, 11 October in the "Ph106 In Box" in East Bridge mailbox.
Remember: only identify your solution set with your class id number, not your name, unless you signed a FERPA waiver.

Variational Approach and Lagrangian Dynamics

Reading

I am teaching the material on introducing Lagrangian dynamics in a different order than Hand and Finch. The material I have discussed in the first week is in §1.1 and §2.1-5. I will be filling in the gaps next week. The calculus of variations is used in many contexts: the textbooks discuss some of these, and others are treated in their problems. I give one example as Problem 4, and you should work through some other examples too. I will not discuss a different way of formulating the variational approach to particle dynamics known as Maupertuis' Principle: this is discussed in the Appendix to Chapter 2 of Hand and Finch which you can read if you are interested.

Problems

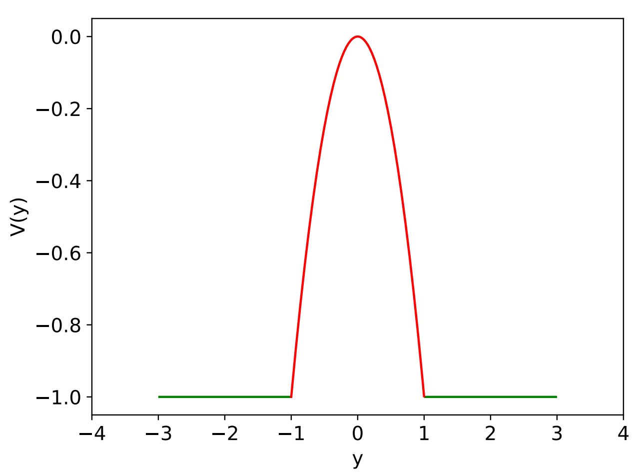
1. **1-D Motion with potential**: Consider a 1-D motion, governed by the equation

$$\frac{1}{2}\dot{y}^2 + V(y) = E \tag{1}$$

with

$$V(y) = \begin{cases} -y^2, & |y| < 1 \\ -1, & |y| >= 1. \end{cases} \tag{2}$$

(a) Sketch the potential energy V.
[Solution]



- **(1 pt)** : Correct graph for $|y| < 1$.
- **(1 pt)** : Correct graph for $|y| >= 1$.

Total sub-points : 2

(b) Suppose we start at rest at $y = \delta$ with $0 < \delta < 1$, compute the amount of time T it takes for the particle to reach $y = 1$. You may find a table of integrals useful for solving the differential equation.

[Solution]

For $0 < y = \delta <= 1$, $V(y) = -y^2$, and the differential equation becomes

$$\frac{1}{2}\dot{y}^2 - y^2 = E \tag{3}$$

- **(1 pt)** : Realising $y <= 1$.
- **(1 pt)** : Substituting $V(y) = -y^2$.

Since the particle starts from **rest** at $y = \delta$, we have $\dot{(y)}(\delta) = 0$ and hence

$$\frac{1}{2}(0)^2 - \delta^2 = E$$

$$E = -\delta^2 \tag{4}$$

- **(1 pt)** : Realising $\dot{y}(\delta) = 0$ (or equivalently the particle is initially at rest).
- **(1 pt)** : Finding E in terms of δ.

(**Read me**: Always read the question carefully and make sure you note down all the initial conditions. Indeed, if you miss out that the particle starts from rest, i.e. $\dot{y}(\delta) = 0$, then you will not be able to show the required results in (c).)

With equation 4, 3 can be written as

$$\frac{1}{2}\dot{y}^2 - y^2 = -\delta^2 \tag{5}$$

Rearranging the terms, we have

$$\begin{aligned} \dot{y^2} - 2y^2 &= -2\delta^2 \\ \dot{y} &= \sqrt{2}\sqrt{y^2 - \delta^2} \\ \frac{dy}{dt} &= \sqrt{2}\sqrt{y^2 - \delta^2} \end{aligned}$$

(Note that we have assumed positive-y as the positive direction of motion when taking the square root on both sides of the equation.)

For $|y| < 1$, and by separation of variables, we have

$$\underbrace{\int_{\delta}^{y} \frac{1}{\sqrt{y'^2 - \delta^2}} dy'}_{(A)} = \underbrace{\sqrt{2}\int_{0}^{t} dt'}_{(B)} \tag{6}$$

- **(1 pt)** : Solving $\dot{y}$ in terms of y and δ.
- **(1 pt)** : Separation of variables **with correct limits**. (-0.5 if the integration variables remain as y and t.)

Supplementary box 1.1: Solving term A.
The following steps are not required for obtaining full credits in this problem.
Using a substitution,

$$\begin{aligned} y' &= \delta \sec\theta \\ dy' &= \delta \sec\theta \tan\theta d\theta \end{aligned}$$

So we can write term A as

$$\int_{\delta}^{y} \frac{1}{\sqrt{y'^2 - \delta^2}} dy' = \int_{\theta(\delta)}^{\theta(y)} \frac{\delta \sec\theta \tan\theta d\theta}{\sqrt{\delta^2 \sec\theta^2 - \delta^2}} = \int_{\theta(\delta)}^{\theta(y)} \frac{\sec\theta \tan\theta d\theta}{\tan\theta} = \int_{\theta(\delta)}^{\theta(y)} \sec\theta d\theta$$

Now, we make another substitution as

$$\begin{aligned} u &= \sec\theta + \tan\theta \\ du &= \sec\theta \tan\theta d\theta + \sec\theta^2 d\theta \\ &= \sec\theta(\sec\theta + \tan\theta) d\theta \\ &= u \sec\theta d\theta \end{aligned}$$

So we have

$$\begin{aligned} \int_{\delta}^{y} \frac{1}{\sqrt{y'^2 - \delta^2}} dy' &= \int_{\theta(\delta)}^{\theta(y)} \sec\theta d\theta \\ &= \int_{u(\theta(\delta))}^{u(\theta(y))} \sec\theta \frac{du}{u \sec\theta} \\ &= \int_{u(\theta(\delta))}^{u(\theta(y))} \sec\theta \frac{du}{u \sec\theta} \\ &= \int_{u(\theta(\delta))}^{u(\theta(y))} \frac{du}{u} \\ &= \Big[\ln|u| \Big]_{u(\theta(\delta))}^{u(\theta(y))} \\ &= \Big[\ln|\sec\theta + \tan\theta| \Big]_{\theta(\delta)}^{\theta(y)} \\ &= \left[\ln\left| \frac{y'}{\delta} + \sqrt{(\frac{y'}{\delta})^2 - 1} \right| \right]_{\delta}^{y} \\ &= \left[\ln\left| \frac{y' + \sqrt{y'^2 - \delta^2}}{\delta} \right| \right]_{\delta}^{y} \\ &= \ln\left| \frac{y + \sqrt{y^2 - \delta^2}}{\delta} \right| - \ln\left| \frac{\delta + \sqrt{\delta^2 - \delta^2}}{\delta} \right| \\ &= \ln\left| \frac{y + \sqrt{y^2 - \delta^2}}{\delta} \right| \end{aligned}$$

With the help of an integral table, you should find that

$$\int \frac{1}{\sqrt{x^2 \pm a^2}} dx = \ln(x + \sqrt{x^2 \pm a^2}) \tag{7}$$

So term A will give you

$$\int_\delta^y \frac{1}{\sqrt{y'^2 - \delta^2}} dy' = \left[\ln \left| y' + \sqrt{y'^2 - \delta^2} \right| \right]_\delta^y = \ln \left| \frac{y + \sqrt{y^2 - \delta^2}}{\delta} \right|$$

- **(1 pt)** : Solving the integral (-0.5 if there are calculation mistakes when substituting the limits into the solution). Note that students might have already substituted $y = 1$ and $t = T$ to the solution. Also give points in this case.

Hence, equation 6 can be written as

$$\ln \left| \frac{y + \sqrt{y^2 - \delta^2}}{\delta} \right| = \sqrt{2} \int_0^t dt' = \sqrt{2} t$$

$$t = \frac{1}{\sqrt{2}} \ln \left| \frac{y + \sqrt{y^2 - \delta^2}}{\delta} \right|$$

For $y = 1$, the required time T will be

$$T = \frac{1}{\sqrt{2}} \ln \left| \frac{1 + \sqrt{1 - \delta^2}}{\delta} \right|$$

- **(1 pt)** : Correct expression for T.

Total sub-points : 8

(c) Show that $T \to +\infty$ when $\delta \to 0$.

[Solution]

From the answer in (b), we can see that

$$\lim_{\delta \to 0} T = \lim_{\delta \to 0} \frac{1}{\sqrt{2}} \ln \left| \frac{1 + \sqrt{1 - \delta^2}}{\delta} \right| = \frac{1}{\sqrt{2}} \lim_{\delta \to 0} \ln \left| \frac{1}{\delta} + \sqrt{\frac{1}{\delta^2} - 1} \right| = \frac{1}{\sqrt{2}} \lim_{\delta \to 0} \ln |\infty| = +\infty$$

Hence $T \to +\infty$ when $\delta \to 0$.

- **(1 pt)** : Proof with reasonable steps (i.e. show that the argument inside the natural logarithm is ∞). **Note**: If students do not include the initial condition $\dot{(y)}(\delta) = 0$ in part (b), and they attempt to do the limiting case, award 0.5 points for their attempt [1].

Supplementary box 1.2: Understanding the mathematics.
Let's think a bit more about what is done here. We go back to part (b), where you are asked to solve the differential equation

$$\frac{1}{2}\dot{y}^2 - y^2 = E$$

[1] The award is explained in Supplementary box 1.2.

In question (b) we impose the initial condition that the particle starts from rest, i.e. $\dot{y}(\delta) = 0$ to obtain $E = -\delta^2$. Indeed, if you miss out this condition, then you will get

$$T = \lim_{\delta \to 0} \frac{1}{\sqrt{2}} \ln \left| \frac{y + \sqrt{y^2 + E}}{\delta + \sqrt{\delta^2 + E}} \right| = \frac{1}{\sqrt{2}} \ln \left| \frac{y + \sqrt{y^2 + E}}{\sqrt{E}} \right|$$

in part (c) which will not absolutely go to ∞ for sure.

The limit $\delta \to 0$ pushes the particle to start from the position where the potential energy $V = 0$. If you just put it here at rest without any initial velocity (imposing the condition $\dot{y}(\delta) = 0$, i.e. the ball is initially at rest), the situation is like you put a ball on the top of the mountain where it is locally flat, and ask how long will it take for the ball to roll down to the bottom ($y = 1$) of the mountain - And obviously, the ball will just stay on the top of the mountain forever and hence it takes **infinite time** ($T \to \infty$) for it to reach the bottom.

However, if the initial velocity is not 0, say $\dot{y}(\delta) = \dot{y}_0$, this means that you are giving the ball an **initial kick** at the top of the mountain and hence the ball will eventually reach the bottom in **finite time**. Hence T will not go to infinity even if $\delta \to 0$.

And of course, intuitively if you put the ball at an initial position $\delta \neq 0$, the potential at this position (which is not zero except for $y = 0$) will make the ball move eventually even if it is initially at rest, and so the time for it to move to the bottom of the mountain (i.e. $y = 1$) is still finite.

Total sub-points : 1

2. **Momentum and Angular Momentum**:

A particle of mass m moves according to the equations

$$x = x_0 + at^2$$
$$y = bt^3$$
$$z = ct$$

Find the angular momentum $\vec{L}$ at any time t. Find the force $\vec{F}$ and from it the torque $\vec{N}$ acting on the particle. Verify that the torque gives the rate of change of angular momentum.

[Solution]

From the equations of the position of the particle, we can get the velocity components of the particle as

$$\dot{x} = \frac{d}{dt}(x_0 + at^2) = 2at \tag{8}$$

$$\dot{y} = \frac{d}{dt}(bt^3) = 3bt^2 \tag{9}$$

$$\dot{z} = \frac{d}{dt}(ct) = c \tag{10}$$

The angular momentum of the particle is just given by $\vec{L} = m\vec{r} \times \vec{v}$, which can be evaluated as:

$$\begin{aligned}
\vec{L} = m\vec{r} \times \vec{v} = m(x\hat{x} + y\hat{y} + z\hat{z}) \times (\dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z}) \\
&= m((x_0 + at^2)\hat{x} + bt^3\hat{y} + ct\hat{z})) \times (2at\hat{x} + 3bt^2\hat{y} + c\hat{z}) \\
&= m \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x_0 + at^2 & bt^3 & ct \\ 2at & 3bt^2 & c \end{vmatrix} \\
&= m\hat{x} \begin{vmatrix} bt^3 & ct \\ 3bt^2 & c \end{vmatrix} - m\hat{y} \begin{vmatrix} x_0 + at^2 & ct \\ 2at & c \end{vmatrix} + m\hat{z} \begin{vmatrix} x_0 + at^2 & bt^3 \\ 2at & 3bt^2 \end{vmatrix} \\
&= m\hat{x}(bct^3 - 3bct^3) - m\hat{y}(cx_0 + act^2 - 2act^2) + m\hat{z}(3bx_0t^2 + 3abt^4 - 2abt^4) \\
&= -2bcmt^3\hat{x} + m(act^2 - cx_0)\hat{y} + m(3bx_0t^2 + abt^4)\hat{z}
\end{aligned}$$

- **(3 pt)** : Correct calculations of the velocity components.
- **(1 pt)** : Stating $\vec{L} = m\vec{r} \times \vec{v}$.
- **(1 pt)** : Correct final expression for the angular momentum.

The acceleration components of the particle are

$$\ddot{x} = \frac{d}{dt}(2at) = 2a \tag{11}$$

$$\ddot{y} = \frac{d}{dt}(3bt^2) = 6bt \tag{12}$$

$$\ddot{z} = \frac{d}{dt}(c) = 0 \tag{13}$$

and hence the force acting on the particle is just

$$\vec{F} = m\vec{a} = m\ddot{x}\hat{x} + m\ddot{y}\hat{y} + m\ddot{z}\hat{z} = 2ma\hat{x} + 6mbt\hat{y}$$

- **(3 pt)** : Correct calculations of the acceleration components.
- **(1 pt)** : Correct final expression for the force (-0.5 if students miss out the mass of the particle m).

The torque acting on the particle is given by $\vec{N} = \vec{r} \times \vec{F}$, and is evaluated as

$$\begin{aligned}
\vec{N} = \vec{r} \times \vec{F} \\
&= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x_0 + at^2 & bt^3 & ct \\ 2am & 6bmt & 0 \end{vmatrix} = \hat{x} \begin{vmatrix} bt^3 & ct \\ 6bmt & 0 \end{vmatrix} - \hat{y} \begin{vmatrix} x_0 + at^2 & ct \\ 2am & 0 \end{vmatrix} + \hat{z} \begin{vmatrix} x_0 + at^2 & bt^3 \\ 2am & 6bmt \end{vmatrix} \\
&= \hat{x}(0 - 6bcmt^2) - \hat{y}(0 - 2acmt) + \hat{z}(6bx_0mt + 6abmt^3 - 2abmt^3) \\
&= -6bcmt^2\hat{x} + 2acmt\hat{y} + (6bx_0mt + 4bmt^3)\hat{z}
\end{aligned}$$

- **(1 pt)** : Stating $\vec{N} = \vec{r} \times \vec{F}$.
- **(1 pt)** : Correct final expression for the torque.

Finally, from the expression for the angular momentum $\vec{L}$, we have

$$\begin{aligned}\frac{d\vec{L}}{dt} &= \frac{d}{dt}(2bcmt^3)\hat{x} + \frac{d}{dt}(m(act^2 - cx_0))\hat{y} + \frac{d}{dt}(m(3bx_0t^2 + abt^4))\hat{z} \\ &= -6bcmt^2\hat{x} + 2acmt\hat{y} + (6bmx_0 + 4abmt^3)\hat{z} \\ &= \vec{N}\end{aligned}$$

which is equal to the torque acting on the particle. Hence this proved that the torque acting on a body is just the rate of change of the angular momentum.

- **(1 pt)** : Differentiating the angular momentum with respect to time (Method).
- **(1 pt)** : The evaluated expression is the same as the torque.

Total sub-points : 13

Supplementary box 1.3: Visualising the motion of the particle.
Let's try to visualise and understand the motion of the particle.
First we consider the x-direction motion. We have

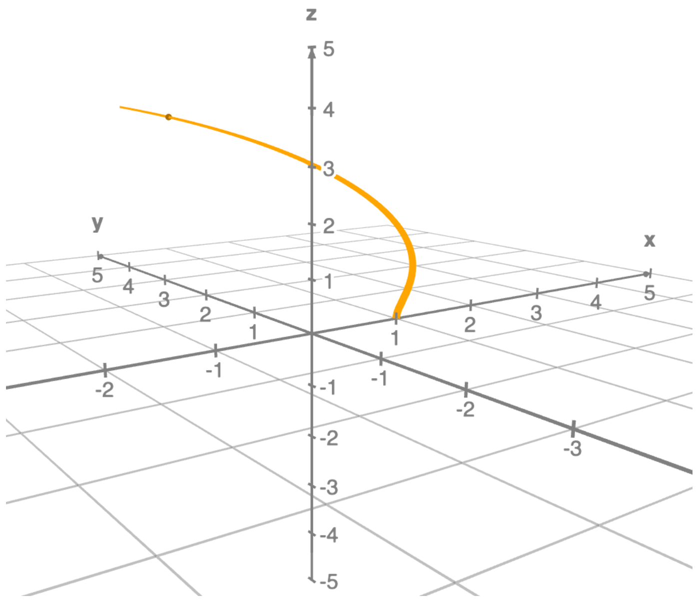
$$\begin{cases} x = x_0 + at^2 \\ \dot{x} = 2at \\ \ddot{x} = 2a \end{cases}$$

The $\ddot{x}$ tells us that the x-acceleration of the particle is a constant. In other words, the x-force acting on the particle is also a constant. This means that the particle is undergoing **uniformly accelerated motion** along the x-direction. One typical example is a particle being thrown up on the surface of the Earth, and it moves under the influence of gravity. Furthermore, the absence of the term proportional to t in the expression for x indicates that the particle has 0 initial velocity along the x-direction.

The motion along the y-direction is a little more complicated because the y-acceleration is **time-dependent**. Indeed, it is increasing with time t. So this means someone could be pulling the particle along the y-direction with an increasing force. Again, the absence of the term proportional to t in the expression for y indicates that the particle has 0 initial velocity along the y-direction.

Finally, the motion along the z-direction is simply a **uniform motion**. It is moving at a z-velocity $= c$ all the time, and the acceleration (and hence the force) along the z-direction is 0 at all times.

Combining the three consideration, you should be able to visualise the 3D-motion of the particle as shown in the figure on the next page (where I have set $x_0 = a = b = c = 1$).



3. **Conservative vs non-conservative forces**:

We explained in class that a conservative force is one for which the work done in moving from one point to another is path-independent. It is equivalent to say that the work done around a closed path vanishes. Stokes' theorem relates the line integral of a vector around a closed path to the integral of the curl of the vector over the area enclosed by the path:

$$\oint_C \vec{F} \cdot d\vec{r} = \int_{A_C} d\vec{a} \cdot \left(\vec{\nabla} \times \vec{F}\right) \tag{14}$$

Thus, any conservative force must have $\vec{\nabla} \times \vec{F} = 0$ everywhere to ensure the left side vanishes for an arbitrary loop C. Calculate the curl to determine which of the following force fields is conservative. For any that are conservative, find the potential energy $U(\vec{r})$.

(a) $F_x = ayz + bx + c$, $F_y = axz + bz$, $F_z = axy + by$. Consider the entire 3-D space.

[Solution]

The curl of the force field is

$$\begin{aligned}
\vec{\nabla} \times \vec{F} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \\
&= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ayz + bx + c & axz + bz & axy + by \end{vmatrix} \\
&= \hat{x}((ax+b) - (ax+b)) - \hat{y}(ay - ay) + \hat{z}(az - az) = \vec{0}
\end{aligned}$$

Since the force field is **curl-less**, it is **conservative**.

By definition we have

$$\begin{cases} -\frac{\partial U}{\partial x} = ayz + bx + c \\ -\frac{\partial U}{\partial y} = axz + bz \\ -\frac{\partial U}{\partial z} = axy + by \end{cases} \tag{15}$$

Integrating equation 15 **from the reference point** $(0,0,0)$ gives

$$\begin{cases} U = -axyz - \frac{bx^2}{2} - cx + f_1(y,z) \\ U = -axyz - byz + f_2(x,z) \\ U = -axyz - byz + f_3(x,y) \end{cases} \tag{16}$$

Comparing the three expressions in equation 16, we get

$$U = -axyz - \frac{bx^2}{2} - cx - byz$$

- **(1 pt)** : Correct calculation of the curl of the force field.
- **(1 pt)** : Attempting to solve the potential function U (Method). Some students may use a different approach in find the potential function (e.g. calculating the work done from a reference point $(0,0,0)$ to an arbitrary point (x,y,z)), which should also be given the credit. Students who do not explicitly state the point of reference, nor giving a integration constant will still get the credit for this problem since the point of reference can be set to make the constant becomes 0.
- **(1 pt)** : Getting the correct expression for the potential function U.

Total sub-points : 3

(b) $F_x = -ze^{-x}$, $F_y = \ln z$, $F_z = e^{-x} + \frac{y}{z}$. Consider only the region of $z > 0$.
[Solution]
The curl of the force field is

$$\begin{aligned} \vec{\nabla} \times \vec{F} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \\ &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -ze^{-x} & \ln z & e^{-x} + \frac{y}{z} \end{vmatrix} \\ &= \hat{x}(\frac{1}{z} - \frac{1}{z}) - \hat{y}(-e^{-x} - (-e^{-x})) + \hat{z}(0) = \vec{0} \end{aligned}$$

Since the force field is **curl-less**, it is **conservative**.
By definition we have

$$\begin{cases} -\frac{\partial U}{\partial x} = -ze^{-x} \\ -\frac{\partial U}{\partial y} = \ln z \\ -\frac{\partial U}{\partial z} = e^{-x} + \frac{y}{z} \end{cases} \tag{17}$$

Integrating equation 23 **from the reference point** $(0,0,1)$ gives

$$\begin{cases} U = -ze^{-x} + f_1(y,z) \\ U = -y\ln z + f_2(x,z) \\ U = -ze^{-x} - y\ln z + 1 + f_3(x,y) \quad \text{(Note that the lower limit is } z = 1.) \end{cases} \tag{18}$$

Comparing the three expressions in equation 29, we get

$$U = -ze^{-x} - y\ln z + e^{-1}$$

Supplementary box 1.3: Why do we need the constraint $z > 0$?
In part (b) you are required to consider only the region $z > 0$. This is because of the term $\ln z$ in the potential function (z has to be larger than 0 in the natural logarithmic function). Otherwise, you will need to modify the potential function as $U = -ze^{-x} - y\ln|z|$.

- **(1 pt)** : Correct calculation of the curl of the force field.
- **(1 pt)** : Attempting to solve the potential function U (Method). -0.5 point if the student does not explicitly state the reference point or include a integration constant in the answer.
- **(1 pt)** : Getting the correct expression for the potential function U.

Total sub-points : 3

(c) $F(\vec{r}) = \vec{h} \times \vec{r}$. Consider the entire 3-D space.
[Solution]
First we write $\vec{F}$ in component form as

$$\vec{F} = \vec{h} \times \vec{r} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ h_x & h_y & h_z \\ x & y & z \end{vmatrix} = (zh_y - yh_z)\hat{x} - (zh_x - xh_z)\hat{y} + (yh_x - xh_y)\hat{z}$$

$$= (zh_y - yh_z)\hat{x} + (xh_z - zh_x)\hat{y} + (yh_x - xh_y)\hat{z}$$

The curl of the force field is

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ zh_y - yh_z & xh_z - zh_x & yh_x - xh_y \end{vmatrix}$$

$$= \hat{x}(h_x - (-h_x)) - \hat{y}(-h_z - h_y) + \hat{z}(h_z - (-h_z))$$

$$= 2h_x\hat{x} + 2h_y\hat{y} + 2h_z\hat{z} \neq \vec{0}$$

Since the force field is **not curl-less**, it is **not conservative**.

Supplementary box 1.4: Einstein summation notation
When we apply Einstein summation notation, repeated index in a single term implies summation of the term over all the values of the index. For instance, the usual position vector $\vec{r}$ can be expressed as

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} = \sum_{i=0}^{3} r_i\hat{e}_i = r_i\hat{e}_i \quad \text{(Summation over index } i \text{ implied)}$$

where $r_1 = x$, $r_2 = y$, $r_3 = z$ and $\hat{e}_i$ are just the usual unit vectors along the x, y and z direction. In terms of Einstein notation, we can find the curl of the force field in this problem in a more elegant way. First note that the cross product $\vec{h} \times \vec{r}$ is just given by

$$\vec{h} \times \vec{r} = h^i r^i \epsilon_{ij}^k \hat{e}_k$$

where $\epsilon_{ij}{}^k$ is called the **Levi-Civita symbol**, defined by:

$$\epsilon_{ij}{}^k = \begin{cases} +1, & \text{if i, j, k are in cyclic permutation of } (1,2,3) \text{ (for 3-D case)} \\ -1, & \text{if i, j, k are in acyclic permutation of } (1,2,3) \text{ (for 3-D case)} \\ 0, & \text{otherwise} \end{cases}$$

For example, $\epsilon_{12}{}^3 = 1$ and $\epsilon_{21}{}^3 = -1$.

Next, the $\vec{\nabla}$ can be written simply as

$$\vec{\nabla} = \frac{\partial}{\partial r^i} = \partial_i$$

With such, we have

$$\begin{aligned}
\vec{\nabla} \times \vec{h} \times \vec{r} &= \vec{\nabla} \times (h^i r^j \epsilon_{ij}{}^k \hat{e}_k) \\
&= \partial_m (h^i r^j \epsilon_{ij}{}^n) \epsilon^m{}_n{}^t \hat{e}_t \\
&= h^i \partial_m r^j \epsilon_{ij}{}^n \epsilon^m{}_n{}^t \hat{e}_t \quad (h^i \text{ is a constant.}) \\
&= h^i \partial_m r^j \epsilon_{ij}{}^n \epsilon^n{}_t{}^m \hat{e}_t \quad (\text{Cyclic rotating } \epsilon^m{}_n{}^t.) \\
&= h^i \partial_m r^j (\delta_{it} \delta_j{}^m - \delta_i{}^m \delta_{jt}) \hat{e}_t \quad (\text{Using } \epsilon_{ijk} \epsilon^{kmn} = \delta_i{}^m \delta_j{}^n - \delta_i{}^n \delta_j{}^m.) \\
&= h^t \partial_j r^j \hat{e}_t - h^i \partial_i r^t \hat{e}_t \\
&= (\partial_j r^j)(h^t \hat{e}_t) - (h^i \partial_i)(r^t \hat{e}_t) \\
&= (\vec{\nabla} \cdot \vec{r}) \vec{h} - (\vec{h} \cdot \vec{\nabla}) \vec{r}
\end{aligned}$$

where the first term consists of the **divergence** of the position vector $\vec{r}$, and the second term is related to the **gradient** of $\vec{r}$. Note that $\vec{\nabla} \cdot \vec{r} = 3$ (**Check this!**) in this case, and $(\vec{h} \cdot \vec{\nabla})\vec{r} = \vec{h}$ (**Check this also!**). So the curl of this force field is

$$\vec{\nabla} \times \vec{h} \times \vec{r} = (\vec{\nabla} \cdot \vec{r})\vec{h} - (\vec{h} \cdot \vec{\nabla})\vec{r} = 3\vec{h} - \vec{h} = 2\vec{h} \neq \vec{0}$$

which is exactly the same as what we have calculated above!

Finally, note that

$$\vec{\nabla} \times \vec{h} \times \vec{r} = (\vec{\nabla} \cdot \vec{r})\vec{h} - (\vec{h} \cdot \vec{\nabla})\vec{r}$$

is just the usual **BAC-CAB** rule for $\vec{A} \times \vec{B} \times \vec{C}$, i.e.

$$\vec{A} \times \vec{B} \times \vec{C} = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

- **(1 pt)** : Correct calculation of the curl of the force field.

Total sub-points : 1

(d) $\mathrm{F}_x = y/(x^2+y^2)$, $F_y = -x/(x^2+y^2)$, $F_z = 0$. [Extra: not for credit; it's tricky. Consider the 3-D space with the line of $x = y = 0$ removed.].

[Solution]

The curl of the force field is

$$\begin{aligned}
\vec{\nabla} \times \vec{F} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \\
&= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{y}{x^2+y^2} & -\frac{x}{x^2+y^2} & 0 \end{vmatrix} \\
&= \hat{x}(0) - \hat{y}(0) + \hat{z}\left[-\left(\frac{(x^2+y^2)(1) - x(2x)}{x^2+y^2} \right) - \frac{(x^2+y^2)(1) - y(2y)}{x^2+y^2} \right] = \vec{0}
\end{aligned}$$

Since the force field is **curl-less**, it is **conservative**.

Supplementary box 1.5: Solving the potential function

It is tricky to solve the potential function here in the sense that the partial differential equations for the potential energy function U with respect to x and y are coupled, i.e.

$$\begin{cases} -\frac{\partial U}{\partial x} = \frac{y}{x^2+y^2} \\ -\frac{\partial U}{\partial y} = -\frac{x}{x^2+y^2} \\ -\frac{\partial U}{\partial z} = 0. \end{cases}$$

One may try to rephrase the problem in terms of cylindrical coordinates (i.e. (r, θ, z)). Note that

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ x^2 + y^2 = r^2 \end{cases}$$

So the force field can be written as

$$\begin{aligned}
\vec{F} &= \frac{y}{x^2+y^2}\hat{x} - \frac{x}{x^2+y^2}\hat{y} \\
&= \frac{r\sin\theta}{r^2}\hat{x} - \frac{r\cos\theta}{r^2}\hat{y} \\
&= -\frac{-\sin\theta\hat{x} + \cos\theta\hat{y}}{r} \\
&= -\frac{\hat{\theta}}{r}
\end{aligned}$$

where $\hat{\theta}$ is just the unit vector along the θ direction in the usual cylindrical coordinate system. In this sense the force field is obviously curl-less. Now, to solve for the potential energy function, by definition we have

$$\begin{cases} -\frac{\partial U}{\partial r} = 0 \\ -\frac{1}{r}\frac{\partial U}{\partial \theta} = -\frac{1}{r} \\ -\frac{\partial U}{\partial z} = 0. \end{cases}$$

Note that in cylindrical coordinate system, the gradient is given by

$$\vec{\nabla} = \frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial \theta} + \frac{\partial}{\partial z}$$

(**Read me**: I encourage readers to proof this as an exercise.)

Integrating **from the reference point** $(r, \theta, z) = (\delta, 0, 0)$ (where $\delta \to 0$), we get

$$\begin{cases} U = f_1(\theta, z) \\ U = \theta \\ U = f_3(r, \theta) \end{cases}$$

Comparing the three expressions, we get

$$U = \theta$$

The only concern is r can never be 0.

- **(1 pt)** : Correct calculation of the curl of the force field.

Total sub-points : 1

4. **Calculus of variations and geodesics**

(a) By extremizing the length of a path $y(x)$ between two points on the (x, y) plane using the calculus of variations, show that the shortest path between two points (the geodesic) on a plane is a straight line. (You don't have to show it is the shortest, only that it is extremal.)

[Solution]

On the x-y plane, the position vector $\vec{r}$ of a point (x, y) is

$$\vec{r} = x\hat{x} + y\hat{y} \tag{19}$$

If you move an infinitesimal distance from this point, the vector change in your position will be

$$d\vec{r} = dx\hat{x} + dy\hat{y} \tag{20}$$

and hence the change in distance (the "path length") is given by the square root of the dot product $d\vec{r} \cdot d\vec{r}$, i.e.

$$ds = \sqrt{d\vec{r} \cdot d\vec{r}} = \sqrt{dx^2 + dy^2} \tag{21}$$

To move from one point (x_1, y_1) to another (x_2, y_2), there are many different paths. For whichever path you follow, to evaluate the total path length, you have to cut the paths into infinitely many short intervals such that the individual path lengths are infinitesimally small as ds. Then, the total path length is just

$$\mathcal{L} = \int ds = \int_{(x_1,y_1)}^{(x_2,y_2)} \sqrt{dx^2 + dy^2} = \int_{x_1}^{x_2} dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \int_{x_1}^{x_2} dx \underbrace{\sqrt{1 + (\dot{y}(x))^2}}_{\text{"Lagrangian"}} \tag{22}$$

where the term in the under-bracket is like the "Lagrangian" L of this problem. Then to find the extremum of this "Lagrangian" L, we make use of the Euler-Lagrange equation

$$\frac{d}{dx}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0 \tag{23}$$

and get

$$\frac{d}{dx}\left(\frac{\partial}{\partial \dot{y}}(\sqrt{1+\dot{y}^2})\right) - \frac{\partial}{\partial y}(\sqrt{1+\dot{y}^2}) = 0$$
$$\frac{d}{dx}\left(\frac{\dot{y}}{\sqrt{1+\dot{y}^2}}\right) = 0$$

which can be integrated easily as

$$\frac{\dot{y}}{\sqrt{1+\dot{y}^2}} = \text{Constant} = C$$

Finally, we have

$$\begin{aligned}
\frac{\dot{y}}{\sqrt{1+\dot{y}^2}} &= C \\
\dot{y} &= C\sqrt{1+\dot{y}^2} \\
\dot{y}^2 &= C^2(1+\dot{y}^2) \\
(1-C^2)\dot{y}^2 &= C^2 \\
\dot{y} &= \frac{C}{sqrt1-C^2} = C' \quad (\leftarrow \text{another constant}) \\
\therefore y &= C'x + D
\end{aligned}$$

where D is yet another constant of integration. Using the boundary condition $y(x_1) = y_1$ and $y(x_2) = y_2$, we obtain

$$y = \frac{y_2 - y_1}{x_2 - x_1}x + \frac{x_2 y_1 - x_1 y_2}{x_2 - x_1} \tag{24}$$

which is an equation of straight line passing through the two end points. Hence, we see that the **geodesic** is a straight line.

Supplementary box 1.6: Going a bit further...
There are two extra exercises I can think of to this problem.

i. **Parametrising x and y**

   Redo the above problem by assuming $x = x(t)$ and $y = y(t)$, i.e, x and y can be parametrised by a common parameter t. With the Euler-Lagrange equation, you should get two differential equations, one for x and one for y. Solving them will give you the parametric form of the equation of straight line.

ii. **Geodesic in 3-D space**

   Find the geodesic between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) in the 3-D cartesian coordinate space.

- **(1 pt)** : Knowing that the infinitesimal distance between two points is $\sqrt{dx^2+dy^2}$.
- **(1 pt)** : Constructing the Lagrangian $L=\sqrt{1+\dot{y}^2}$ (or $\sqrt{\dot{x}^2+1}$).
- **(1 pt)** : Using and simplifying the Euler-Lagrange equation.
- **(1 pt)** : Showing that $\frac{\dot{y}}{\sqrt{1+\dot{y}^2}}$ is a constant.
- **(1 pt)** : Showing that $\dot{y}$ is also a constant.
- **(1 pt)** : Solving $y(x)$ in terms of x and constants.
- **(1 pt)** : Using the boundary conditions and solving the constants.

Total sub-points : 7

(b) For the geodesic on a sphere, use spherical polar coordinates θ, ϕ. Derive the expression for the length of a path $\phi(\theta)$ between two fixed points on the unit sphere. Use the calculus of variations to find a differential equation for ϕ that specifies the shortest path. Work the problem far enough to find a first order differential equation for $d\phi/d\theta = \ldots$ (Again, you don't have to show it is the shortest, only that it is extremal. This works for both the shortest *and* the longest paths; can you picture both?)

Extra (not for credit): It is somewhat tricky calculus/algebra to integrate the equation for $d\phi/d\theta$, and then to show the resulting expression defines a great circle — the intersection of the plane containing the two points and the center with the surface of the sphere. You might at least like to show that one solution is $d\phi/d\theta = 0$ i.e. $\phi =$ constant, the special case of a great circle through the two poles. The problem is harder to solve if we write the path as $\theta(\phi)$ rather than $\phi(\theta)$: do you see why this is so?

[Solution]

In terms of spherical coordinates, the position vector $\vec{r}$ of a point (r, θ, ϕ) is

$$\vec{r} = r\sin\theta\cos\phi\hat{x} + r\sin\theta\sin\phi\hat{y} + r\cos\theta\hat{z} \tag{25}$$

Since you are constrained to move on a sphere (let's keep $r = R$ to make the derivation general for the time being, although the question states that you are on a **unit** sphere, in which case $r = 1$), $r = R$ =constant and we have

$$\vec{r} = R\sin\theta\cos\phi\hat{x} + R\sin\theta\sin\phi\hat{y} + R\cos\theta\hat{z} \tag{26}$$

If you move an infinitesimal distance from this point, the vector change in your position will be

$$d\vec{r} = R(d\theta\cos\theta\cos\phi - d\phi\sin\theta\sin\phi)\hat{x} + R(d\theta\cos\theta\sin\phi + d\phi\sin\theta\cos\phi)\hat{y} - Rd\theta\sin\theta\hat{z} \tag{27}$$

and hence the change in distance (the "path length") is given by the square root of the dot product $d\vec{r}\cdot d\vec{r}$, i.e.

$$\begin{aligned} ds &= \sqrt{d\vec{r}\cdot d\vec{r}} \\ &= \sqrt{R^2(d\theta\cos\theta\cos\phi - d\phi\sin\theta\sin\phi)^2 + R^2(d\theta\cos\theta\sin\phi + d\phi\sin\theta\cos\phi)^2 + R^2(d\theta\sin\theta)^2} \\ &= R\sqrt{d\theta^2 + \sin^2\theta d\phi^2} \end{aligned}$$

To move from one point (θ_1, ϕ_1) to another (θ_2, ϕ_2), there are many different paths. For whichever path you follow, to evaluate the total path length, you have to cut the paths into infinitely many short intervals such that the individual path lengths are infinitesimally small as ds. Then, the total path length is just

$$\mathcal{L} = \int ds = \int_{(\theta_1,\phi_1)}^{(\theta_2,\phi_2)} R\sqrt{d\theta^2 + \sin^2\theta d\phi^2} = R\int_{\theta_1}^{\theta_2} d\theta \underbrace{\sqrt{1+\sin^2\theta(\dot{\phi}(\theta))^2}}_{\text{“Lagrangian”}} \tag{28}$$

where the term in the under-bracket is like the “Lagrangian” L of this problem. Then to find the extremum of this “Lagrangian” L, we make use of the Euler-Lagrange equation

$$\frac{d}{d\theta}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0 \tag{29}$$

and get

$$\begin{aligned} \frac{d}{d\theta}\left(\frac{\partial}{\partial \dot{\phi}}(\sqrt{1+\dot{\phi}^2\sin^2\theta})\right) - \frac{\partial}{\partial \phi}(\sqrt{1+\dot{\phi}^2\sin^2\theta}) &= 0 \\ \frac{d}{d\theta}\left(\frac{\dot{\phi}\sin^2\theta}{\sqrt{1+\dot{\phi}^2\sin^2\theta}}\right) &= 0 \end{aligned}$$

which can be integrated easily as

$$\frac{\dot{\phi}\sin^2\theta}{\sqrt{1+\dot{\phi}^2\sin^2\theta}} = \text{Constant} = C$$

Then, we have

$$\begin{aligned} \frac{\dot{\phi}\sin^2\theta}{\sqrt{1+\dot{\phi}^2\sin^2\theta}} &= C \\ \dot{\phi}\sin^2\theta &= C\sqrt{1+\dot{\phi}^2\sin^2\theta} \\ \dot{\phi}^2\sin^4\theta &= C^2(1+\dot{\phi}^2\sin^2\theta) \\ (\sin^4\theta - C^2\sin^2\theta)\dot{\phi}^2 &= C^2 \\ \dot{\phi} = \frac{d\phi}{d\theta} &= \frac{C}{\sin\theta\sqrt{\sin^2\theta - C^2}}. \end{aligned}$$

Supplementary box 1.7: Solving the special case...
As suggested by the problem set, let's solve the special case where $\frac{d\phi}{d\theta} = 0$. We have

$$\begin{aligned} &\dot{\phi} = \frac{d\phi}{d\theta} = 0 \\ &\int_{\phi_0}^{\phi} d\phi' = \text{Constant} = C \\ &\phi - \phi_0 = C \end{aligned}$$

Since $\phi = \phi_0$ initially, we have $C = 0$. Applying the sine operation to both sides, and multiplying both sides by $R \sin\theta$, we get

$$\begin{aligned} R\sin(\phi - \phi_0)\sin\theta &= 0 \\ R\sin\phi\sin\theta\cos\phi_0 - R\cos\phi\sin\theta\sin\phi_0 &= 0 \\ y\cos\phi_0 - x\sin\phi_0 &= 0 \end{aligned}$$

which is the equation of the (vertical) plane with the z-axis lying on it. Intersecting this plane with the sphere, the intersecting curve defines the great circle, which is the geodesic, passing through the two poles on the sphere.

Supplementary box 1.8: Doing the inverse problem...[a]

Now let's find the differential equation that describes a big circle on the sphere. A big circle is the intersection of a plane containing the origin and a unit sphere. This means that we can get its equation by substituting

$$x = \sin\theta\cos\phi, \quad y = \sin\theta\sin\phi, \quad z = \cos\theta \tag{30}$$

into the equation of the plane

$$Ax + By + Dz = 0 \tag{31}$$

This gives the relation

$$\cos(\phi + \alpha) = C\cot\theta \tag{32}$$

where we devided both sides of Eq. (31) by $\sqrt{A^2+B^2}$ and introduced $C \equiv -D/\sqrt{A^2+B^2}$ and $\cos\alpha \equiv \frac{A}{\sqrt{A^2+B^2}}$.

Now let's differentiate this equation. We get

$$\begin{aligned} \sin(\phi+\alpha)\phi' = \frac{C}{\sin^2\theta} &\Rightarrow \sqrt{1 - \frac{C^2}{\tan^2\theta}}\phi' = \frac{C}{\sin^2\theta} \\ \phi' &= \frac{C}{\sqrt{1+C^2}} \frac{1}{\sin\theta\sqrt{\sin^2\theta - \frac{C^2}{1+C^2}}} \end{aligned} \tag{33}$$

Making the identification const $= \frac{C}{\sqrt{1+C^2}}$ we see that the differential equations coincide. Thus any two solutions coincide as well up to an additive constant. Adding a constant to ϕ just maps a big circle to another big circle, so this does not affect the conclusion that big circles are geodesics on a sphere.

[a] Credit to Professor Alan Weinstein.

Supplementary box 1.9: Solving the exact differential equation

Finally, we'll attempt to solve the real differential equation, i.e.

$$\int_{\phi_0}^{\phi} d\phi' = \int \frac{C}{\sin\theta'\sqrt{\sin^2\theta' - C^2}} d\theta'$$

This requires a tricky substitution:

$$u = C \cot \theta'$$
$$du = -C \csc^2 \theta' d\theta'$$

Then, the differential equation becomes

$$\begin{aligned}
\phi - \phi_0 &= \int \frac{C}{\sin \theta' \sqrt{1 - C^2 \sin^2 \theta'}} \frac{du}{(-C \csc^2 \theta)} \\
&= -\int \frac{\sin \theta' du}{\sin \theta' \sqrt{1 - C^2 \csc^2 \theta'}} \\
&= -\int \frac{du}{\sqrt{1 - (C^2 + C^2 \cot^2 \theta')}} \\
&= -\int \frac{du}{\sqrt{1 - C^2 - u^2}}
\end{aligned}$$

Now, we need another substitution:

$$u = \sqrt{1 - C^2} \sin \alpha$$
$$du = \sqrt{1 - C^2} \cos \alpha d\alpha$$

Then, we get

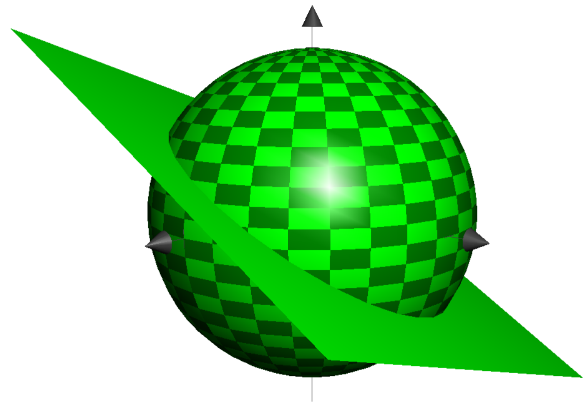
$$\begin{aligned}
\phi - \phi_0 &= -\int \frac{du}{\sqrt{1 - C^2 - u^2}} \\
&= -\int \frac{\sqrt{1 - C^2} \cos \alpha d\alpha}{\sqrt{(1 - C^2) - (1 - C^2) \sin^2 \alpha}} \\
&= -\int \frac{\cos \alpha d\alpha}{\cos \alpha} \\
&= -\int d\alpha = -\alpha = -\sin^{-1} \left(\frac{u}{\sqrt{1 - C^2}} \right) = -\sin^{-1} \left(\frac{C \cot \theta}{\sqrt{1 - C^2}} \right)
\end{aligned}$$

Hence we have

$$\begin{aligned}
\phi - \phi_0 &= -\sin^{-1} \left(\frac{C \cot \theta}{\sqrt{1 - C^2}} \right) \\
\sin -(\phi - \phi_0) &= \frac{C \cot \theta}{\sqrt{1 - C^2}} \\
-\sin (\phi - \phi_0) &= \frac{C}{\sqrt{1 - C^2}} \frac{\cos \theta}{\sin \theta} \\
-\sin \phi \cos \phi_0 + \cos \phi \sin \phi_0 &= \frac{C}{\sqrt{1 - C^2}} \frac{\cos \theta}{\sin \theta} \\
-R \sin \theta \sin \phi \cos \phi_0 + R \sin \theta \cos \phi \sin \phi_0 &= \frac{C}{\sqrt{1 - C^2}} (R \cos \theta) \\
-y \cos \phi_0 + x \sin \phi_0 &= \frac{C}{\sqrt{1 - C^2}} z = C' z
\end{aligned}$$

which is the equation of plane passing through the two end points and the centre of the sphere (where R is the radius of the sphere).

By intersecting this plane equation with the equation of the sphere, we obtain the intersection of the two surface, which is a great circle on the sphere (See the figure below).



- **(1 pt)** : Knowing that the infinitesimal distance between two points is $R\sqrt{d\theta^2 + d\phi^2 \sin^2\theta}$.
- **(1 pt)** : Constructing the Lagrangian $L = \sqrt{1 + \dot{\phi}^2 \sin^2\theta}$.
- **(1 pt)** : Using and simplifying the Euler-Lagrange equation.
- **(1 pt)** : Showing that $\frac{\dot{\phi}\sin^2\theta}{\sqrt{1+\dot{\phi}^2\sin^2\theta}}$ is a constant.
- **(1 pt)** : Getting the correct expression for $\frac{d\phi}{d\theta}$.

Total sub-points : 5

5. **Particle constrained to elliptical wire**

In this problem you use the Lagrangian approach to find the equation of motion for a particle of mass m sliding on a frictionless vertical elliptical wire, with horizontal principal axis a and vertical principal axis b, in gravity g. Use the angle α giving the Cartesian coordinates of the mass as $x(t) = a\cos\alpha(t), y(t) = b\sin\alpha(t)$ to describe the motion.

(a) Verify that with the given parametrization, the bead lies on the ellipse for all time, so that the constrained dynamics can be described by the single variable $\alpha(t)$.

[Solution]

Note that

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \cos^2\alpha + \sin^2\alpha = 1$$
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

which is an equation of an ellipse. This shows that at any time t, the position of the particle (x, y) must satisfy the above ellipse equation, i.e. the particle must lie on the ellipse at all times. Hence the parametrization is justified.

- **(1 pt)** : Attempting to connect x and y by eliminating the parameter α (Method).
- **(1 pt)** : Obtaining the ellipse equation.

Total sub-points : 2

(b) Now derive the differential equation of motion for α.

[Solution]

The velocity components of the particle are given by

$$\begin{cases} \dot{x} = -a\dot{\alpha}\sin\alpha \\ \dot{y} = b\dot{\alpha}\cos\alpha \end{cases} \tag{34}$$

Then the kinetic energy of the particle is

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m(a^2\dot{\alpha}^2\sin^2\alpha + b^2\dot{\alpha}^2\cos^2\alpha) = \frac{1}{2}m\dot{\alpha}^2(a^2\sin^2\alpha + b^2\cos^2\alpha) \tag{35}$$

The (gravitational) potential energy of the particle is

$$V = mgy = mgb\sin\alpha \tag{36}$$

Hence the Lagrangian for the particle is

$$L = T - V = \frac{1}{2}m\dot{\alpha}^2(a^2\sin^2\alpha + b^2\cos^2\alpha) - mgb\sin\alpha \tag{37}$$

Using the Euler-Lagrange equation, we get

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\alpha}}\right) - \frac{\partial L}{\partial \alpha} = 0$$

$$\frac{d}{dt}\left[m\dot{\alpha}(a^2\sin^2\alpha + b^2\cos^2\alpha)\right] - \left[\frac{1}{2}m\dot{\alpha}^2(2a^2\sin\alpha\cos\alpha - 2b^2\sin\alpha\cos\alpha) - mgb\cos\alpha\right] = 0$$

$$m\ddot{\alpha}(a^2\sin^2\alpha + b^2\cos^2\alpha) + m\dot{\alpha}(2a^2\dot{\alpha}\sin\alpha\cos\alpha - 2b^2\dot{\alpha}\cos\alpha\sin\alpha)$$
$$-m\dot{\alpha}^2\sin\alpha\cos\alpha(a^2 - b^2) + mgb\cos\alpha = 0$$

$$\underbrace{m\ddot{\alpha}(a^2\sin^2\alpha + b^2\cos^2\alpha)}_{\propto\ddot{\alpha}} + \underbrace{m\dot{\alpha}^2\sin\alpha\cos\alpha(a^2 - b^2)}_{\propto\dot{\alpha}^2} + mgb\cos\alpha = 0$$

- **(1 pt)** : Obtaining the velocity components of the particle (can be merged into the next point).
- **(1 pt)** : Evaluating the kinetic energy of the particle.
- **(1 pt)** : Writing down the potential energy of the particle.
- **(1 pt)** : Writing down the Lagrangian of the problem.
- **(2 pt)** : .Using and simplifying the Euler-Lagrange equation.

Total sub-points : 5

(c) You should find a term in the equation of motion that is proportional to $\dot{\alpha}^2$, as well as the "obvious" $\ddot{\alpha}$ acceleration term. Give an intuitive interpretation of the $\dot{\alpha}^2$ acceleration term, including why this term disappears in the circular limit $a = b$ and thinking of the constraint forces of the wire on the bead.

[Solution][2]

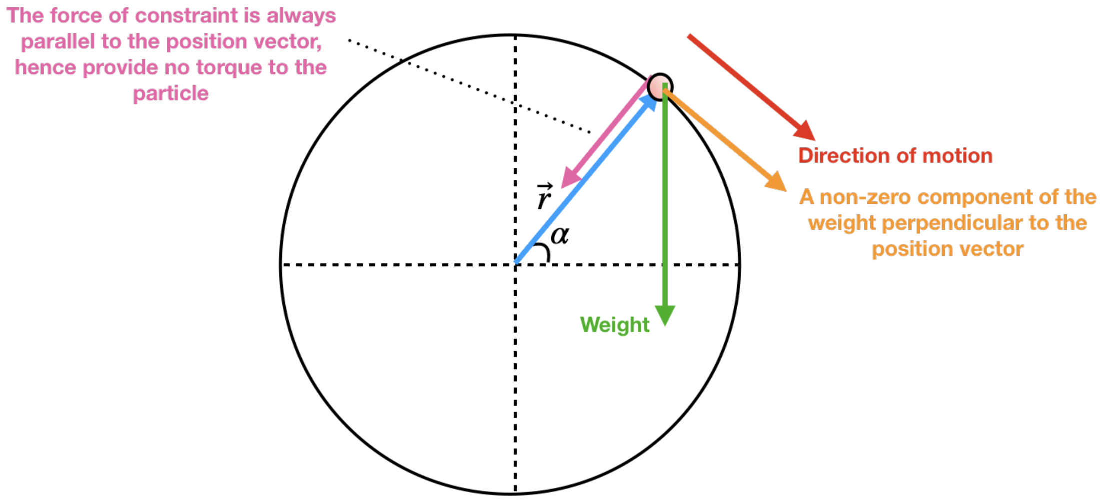
The equation in part (b) can be understood as the Newton's law for the acceleration resolved along the wire. The second term, proportional to $\dot{\alpha}^2$, vanishes for circular motion $a = b$, and in the case the equation is simply understood as the component of

[2]Credit for part of the solutions from Professor Alan Weinstein.

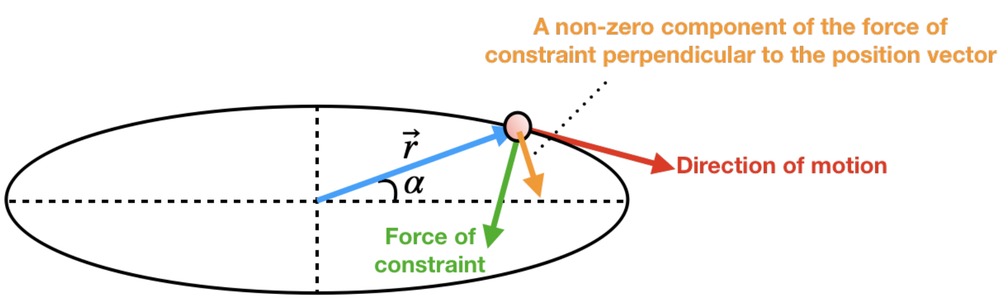
the gravitational force along the wire giving the acceleration along the wire. In other words,

$$\begin{aligned} ma^2\ddot{\alpha} = mR^2\ddot{\theta} &= \text{Rate of change of angular momentum of the particle} \\ &= mga\cos\alpha \\ &= \text{Torque acting on the particle by gravitational force} \end{aligned}$$

But we know that circular motion also has radial acceleration: what about the force needed for this component? The answer is that the "extra" force needed beyond the $mg\sin\alpha$ from the gravitational force is supplied by the force of constraint, which is perpendicular to the wire and so radial. **This explains why the term $\propto \dot{\alpha}^2$ vanishes in the case when $a = b$: The force of constraint (i.e. the normal force acting on the particle by the wire) is always parallel to the position vector of the particle in the circular wire case, and hence acts no torque on the particle**!



For the ellipse, although the force of constraint from the wire, i.e. the normal reaction from the wire, is always perpendicular to the direction of motion of the particle, it is *not* always parallel to the position vector $\vec{r}$ of the particle, and hence it will also contribute torque to the particle, apart from the component of the gravitational force!



Mathematically (and intuitively), we have

$$\underbrace{m\ddot{\alpha}(a^2\sin^2\alpha + b^2\cos^2\alpha)}_{\text{rate of change of angular momentum}} = \underbrace{mgb\cos\alpha}_{\text{torque by weight}} \underbrace{- m\dot{\alpha}^2\sin\alpha\cos\alpha(a^2-b^2)}_{\text{torque by force of constraint}} \tag{38}$$

Equivalently, the extra contribution of torque from the force of constraint causes the emergence of the non-vanishing $\dot{\alpha}^2$ term: the acceleration from the $\dot{\alpha}$ term is *not* always perpendicular to the wire (in which case it could be produced by a perpendicular constraint force), and so some part of the gravitational force must produce the component of the $\dot{\alpha}^2$ acceleration along the wire. More quantitatively, the acceleration when the mass is at $\vec{r}$ is, by double differentiation,

$$\ddot{\vec{r}} = \sqrt{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha}\, \ddot{\alpha}\hat{t} - \dot{\alpha}^2 \vec{r} \tag{39}$$

with $\hat{t}$ the tangent vector to the ellipse at that point. For the ellipse the radius vector $\vec{r}$ is not perpendicular to the tangent vector, and so the second term on the right hand side of Eq. (39) contributes to the tangential acceleration equation.

Supplementary box 1.10: Small angle approximation
Consider the equation you have obtained in part (b). Suppose $a = b$ and let's assume α is small. Give an approximation to the equation and solve for α. You should find that α is uniformly accelerating. Visualise the motion of the particle in this case.

- **(1 pt)** : Trying to explain that the $\dot{\alpha}^2$ term is related to the force of constraint.
- **(1 pt)** : Explaining why the $\dot{\alpha}^2$ term vanishes in the $a = b$ case.
- **(1 pt)** : Relating $mgb \cos \alpha$ and the $\ddot{\alpha}$ terms to the torque acting on and rate of change of angular momentum of the particle (Also accept stating the equation as the "torque" version of Newton's second law) .
- **(1 pt)** : Explaining why the $\dot{\alpha}^2$ term do not vanish in the $a \neq b$ case (Force of constraint is not always perpendicular to the position vector, hence provides torque to the particle).

Total sub-points : 4

Maximum points obtainable for this problem set: 55 pt