Physics 106a, Classical Mechanics
Mid-term Exam, Fall Term, 2019

Do not read beyond this page until you are ready to start the exam!

Material: The exam covers the material of Lectures 1-7, up to "Driven, Damped Oscillations".

Due date: Due in Ph106 box, East Bridge, on Friday 7pm, November 1, 2019. Late exams will require extenuating circumstances; otherwise, no credit will be given. The "one free extension" *cannot* be used on the mid-term.

Logistics: The exam consists of four problems, on pages 2 to 5 of this file. Do not look at the problems until you are ready to start it. Please use a blue book for the exam. Typed exams on individual pages are allowed, but all the typing must be completed in the allotted time. There are **four** questions.

Time: Four hours maximum. You may take breaks, not counted in the 4 hours; but please limit the time spent on breaks. Of course, you should not be consulting references or working on the problems during the breaks. You are under the honor code!

Resources: The midterm and final are not collaborative. All questions must be done on your own, without consulting anyone else. You may consult your own notes (both in-class and notes on this class you or a colleague in the class have made), the text by Hand and Finch or by Taylor, and lecture notes and solution sets on the Ph 106a website. You may not consult any other material, including other textbooks, the web (except for the current Ph106a website), material from previous years' Ph106 or any other classes, or copies you have made of such material, or any other resources. Calculators and symbolic manipulation programs are not allowed or needed.
If these instruction are not clear, please consult with me before starting the exam.

Grading: Points for each part of each problem are noted in square brackets, e.g. [2 points]. The exam counts for approximately 15% of the total term grade. Midterm grades reported to the registrar will be P/F. You will get a P if you handed in A1, A2, A3 and this midterm; else, F.

Reminder: Due date: Friday 7pm, 1 November, 2019

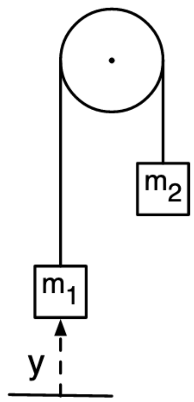
Problem 1 - Miscellaneous questions

This problem consists of 5 independent short problems.

(a) **Spring pendulum**

[5 points] A pendulum is made from a mass m hanging in gravity g on a spring with the other end fixed. The undistorted length of the spring is l and the spring constant is k. Find the Lagrangian for the full three dimensional motion using spherical polar coordinates r, θ, ϕ with the origin at the fixed end of the spring. What are the constants of the motion, if any?

(b) **Pulley with friction**



[4 points] Two masses m_1, m_2 are connected by a massless rope running over a pulley with moment of inertia I and radius a rotating about a fixed axle. There is no slip between the rope and the pulley, so that the pulley rotates when the masses move. However due to friction in the axle, the pulley requires a torque (force times moment arm) $\eta\omega$ with ω the rotation rate. Find the virtual work for a virtual displacement δy when the mass m_1 is at position y and moving upwards with speed $\dot{y}$. What is the generalized force $\mathcal{F}_y$?

(c) **Bead on wire:**

[4 points] A bead of mass m slides in gravity g on a frictionless wire in a vertical plane with shape described by the equation $x^3 + xz + z^3 = 1$ (with x horizontal, z vertical). Find the complete set of equations that could be analyzed (e.g. put on a computer) to give the motion.

(d) **The Hamiltonian:**

[3 points] The kinetic and potential energies of a system with a single degree of freedom characterized by the generalized coordinate q are

$$T = \frac{1}{2}m\dot{q}^2 + \frac{1}{2}m\omega^2 \sin^2 \alpha\, q^2, \quad V = mg\cos\alpha\, q, \tag{1}$$

with m, g, α, ω constants. What is the Hamiltonian as a function of coordinate and momentum (not velocity)?

(e) **Driven oscillator:**

[5 points] A simple harmonic oscillator is described by the equation of motion

$$\ddot{q} + q = F(t), \tag{2}$$

with $F(t)$ being the driving force, and is initially at rest at $q = 0$. Find the solution for $q(t)$ for $t > 0$ if an oscillating force $F(t) = \sin(2t)$ is switched on at time $t = 0$.

Problem 2 - Simple harmonic oscillator with time-dependent Lagrangian

Assume the Lagrangian for a certain one-dimensional motion is given by

$$L = e^{\phi t}(\frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2) \tag{3}$$

where ϕ, m, and k are positive constants.

(a) **[2 points]** Write down the Euler-Lagrange equation. Then apply the equation to the Lagrangian and obtain the equation of motion.

(b) **[8 points]** Solve the equation by trying q in exponential form e^{At}, and discuss the different solutions in different conditions.

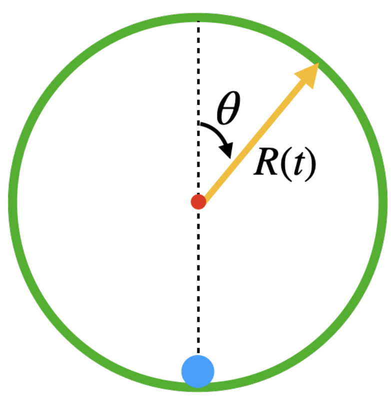
(c) **[8 points]** Suppose a point transformation is made to another generalized coordinate, given by $S = e^{\frac{\gamma t}{2}}q$. What is the Lagrangian in terms of S? Find Lagrange's equation, and possible constants of motion. Describe the relationship of new solutions with results of part (b).

Problem 3 - Particle moving on a circular track with time-varying radius

A point particle of mass m is set to move inside a vertical circular track of radius $R(t)$. This is a strange circular track because its radius changes with time, in particular the radius is of the form

$$R(t) = R_0\left(1 + \frac{a}{2}\sin t\right), \tag{4}$$

where R_0 and a are unspecified constants such that $R > 0$ for all t. Initially the particle at the bottom of the track. It is then given a kick at time $t = 0$ such that it has a certain initial angular velocity ω_0. In this problem, use polar coordinates (r, θ) to describe the position of the particle. Take the acceleration due to gravity (which is acting downward) as g.



(a) **[2 points]** Write down the Lagrangian for the particle in terms of the generalized coordinates (r, θ). Do not impose any constraints to the Lagrangian yet.

(b) **[3 points]** What is the constraint in this problem? Choose the right word(s) from {*holonomic, non-holonomic, rheonomous (depends explicitly on time)* and *scleronomous (does not depend explicitly on time*} to describe the constraint.

(c) **[3 points]** Apply the Euler-Lagrange equation with Lagrange multipliers to find the equation of motion for r and θ.

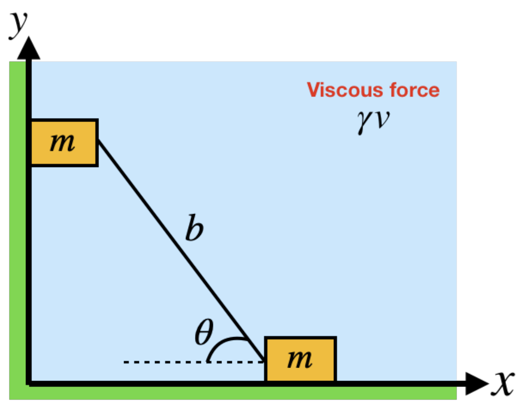
(d) [**5 points**] Using your constraint in (b), further reduce your equations in (c) to a pair of equations for θ and λ only (i.e. eliminate r from your equations). You are *NOT* required to solve the resulting equations.

(e) [**5 points**] Let's turn the circular track back to a normal track, i.e. let's set $a = 0$. Find the solution for $\dot{\theta}$ in terms of θ *starting from the Euler-Lagrange equation.* (Hint: Consider the 2^{nd} equation. Find an integrating factor that makes the equation into an exact differential.)

(f) [**2 points**] Hence find the magnitude of the force of constraint in terms of θ and θ_0.

(g) [**3 points**] Finally, if we want the ball to be able to do a *complete round trip* around the circular track, what is the minimum value of ω_0?

Problem 4 - Virtual work, generalized force and equation of motion



As shown in the figure, a system consists of two identical blocks of mass m connected by a massless rigid rod of length b with rotatable joints at the two blocks. The whole system is placed in a viscous liquid (maybe it's water). Assume that the viscous force only acts along the direction of motion of the individual masses, and has a magnitude of $f = \gamma v$, where v is the velocity of the individual masses and γ is a constant. The angle of elevation of the left mass from the right mass is θ. Take the acceleration due to gravity, which acts in the negative y direction, as g. We want to investigate the motion of the system. Assume motions only take place in the the x-y plane (i.e. There is no motion along the z direction). Neglect friction with respect to the motion of the rod.

(a) [**2 points**] How many constraint(s) is(are) there in this problem? Is the constraint holonomic or non-holononic?

(b) [**1 point**] Assume that the left mass can only move vertically and the right mass can only move horizontally. How many degree(s) of freedom do(es) this system have?

(c) [**2 points**] Draw a free body diagram for the blocks in the system. Remember you also have to consider the internal forces.

(d) [**1 point**] Upon a virtual displacement, which forces in (c) can do a virtual work on the system of blocks?

(e) [**4 points**] Let's denote the position of the left and right mass as $(0, y)$ and $(x, 0)$ respectively. Find the virtual work upon a small displacement for the system in terms of δy (for the left mass) and δx (for the right mass). Recall that x and y are *NOT* independent coordinates. In the view of this, express the virtual work in terms of $\delta\theta$ and θ only.
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(f) **[3 points]** What is(are) a (set of) suitable generalized coordinate(s) for this system? Does this agree with your answer in (b)? Also identify the generalized force for this problem in terms of the generalized coordinates you have chosen.

(g) **[2 points]** Using D'Alembert's principle or the Golden Rule #1 and show that the equation of motion is in the form

$$f_1(m, b, g, \gamma, \theta)\ddot{\theta} + f_2(m, b, g, \gamma, \theta)\dot{\theta} + f_3(m, b, g, \gamma, \theta) = 0 \tag{5}$$

where f_1, f_2 and f_3 are undetermined functions of m, b, g, γ and θ.

(h) **[1 point]** If $\gamma = 0$, what is the equation of motion?

(i) **[10 points]** Formulate the equation of motion in part (h) (i.e. $\gamma = 0$) starting from the generalized coordinates (x, y). Apply the Lagrange's equations with Lagrange multipliers. Express the equations in terms of θ and solve for $\dot{\theta}^2$ in terms of θ. Finally, find the force of constraints in terms of θ and θ_0. Assume that $\theta = \theta_0$ initially, and the system starts from rest.