Physics 106a, Caltech — 22 October, 2019

Lectures 7 & 8: Driven, Damped Oscillations

Driven Oscillator

Now we turn to the driven linear harmonic oscillator described by the equation (after scaling out constants)

\[ \ddot{q} + q = F(t). \]  

(1)

This equation is inhomogeneous but still linear in \( q \). This means that for a forcing function that is a linear combination of two forcing functions, the inhomogeneous solution is the same linear superposition of the individual inhomogeneous solutions

\[ F_1(t) \Rightarrow q_1(t), \quad F_2(t) \Rightarrow q_2(t) \quad \text{then} \quad F(t) = \alpha F_1(t) + \beta F_2(t) \Rightarrow q(t) = \alpha q_1(t) + \beta q_2(t). \]  

(2)

Let’s look at step, impulse, and oscillatory forcing functions. Superposition means that the solution of any one of these can be derived from the others, and that each one can be used to construct the solution to an arbitrary \( F(t) \).

The general solution of Eq. (1) is

\[ q(t) = q_s(t) + q_t(t) \]  

(3)

where

- \( q_s(t) \) is a solution of the inhomogeneous equation (1) — the particular integral, or the “steady state” solution;
- \( q_t(t) \) is the general solution to the homogeneous equation (Eq. (1) with \( F(t) = 0 \)) with two integration constants) — the complementary function, or the “transient” solution. The solution is

\[ q_t(t) = a \cos t + b \sin t, \quad \text{or} \quad Ae^{it} + Be^{-it}, \]  

(4)

with \( a, b \) or \( A, B \) the two constants to be fixed from the initial conditions. The solution could also be written \( q_t(t) = a \cos(t + \phi) \) with \( a, \phi \) the two constants.

Thus Eq. (3) has two unknown constants that are set by initial conditions. This is called the “transient” solution because in most realistic problems, there is some damping or dissipation (discussed below). If we added dissipation to the equation, \( q_t(t) \) would decay exponentially. For example, a pendulum would come to rest before any forcing. We’ll assume that the forcing is zero for \( t < 0 \) and that this implies \( q(t) = 0 \) for \( t < 0 \), so that the initial conditions are \( q(0) = \dot{q}(0) = 0 \).

You should make sure you can arrive at the following solutions:

**Unit step function:** \( F(t) = \Theta(t) \) with \( \Theta(t) \) the Heaviside function

\[ \Theta(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ 1 & \text{for } t > 0 \end{cases} \]  

(5)

The solution for \( t > 0 \) is

\[ q = 1 - \cos t. \]  

(6)

Check that \( q(0) = \dot{q}(0) = 0 \), as required to match to \( q(t < 0) \).
**Unit impulse:** $F(t) = \delta(t)$ with $\delta(t)$ the Dirac delta function defined by its integral properties

$$\int_{t'=a}^{t'+b} g(t')\delta(t-t')dt' = g(t') \quad (7)$$

for any smooth $g(t)$ and any $a, b > 0$. The function $\delta(t-t')$ (actually a generalized function) is infinitely high and infinitely narrow, but with a unit integral. Recall that the Heaviside function is the integral of the delta function, and the delta function is the derivative of the Heaviside function.

The solution for $t > 0$ is (remember I am assuming $q(t) = 0$ for $t \leq 0$)

$$q(t) = \sin t. \quad (8)$$

We can argue this physically: a unit impulse gives unit momentum change, so the speed $\dot{q}$ for our unit mass “particle” after the impulse is $\dot{q}(t = 0^+) = 1$, whilst $q(0^+)$ remains zero. Or we can integrate the equation of motion formally from $t = -\epsilon$ to $t = \epsilon$ with $\epsilon \to 0$.

The solution for a unit impulse at time $t'$ (i.e., $F(t) = \delta(t-t')$) is called the Green’s function, and is (by a simple time translation)

$$G(t, t') = \begin{cases} \sin(t-t') & \text{for } t > t' \\ 0 & \text{for } t < t' \end{cases} \quad (9)$$

assuming no response before the impulse is applied (the casual Green’s function).

By superposition, the solution for a general forcing $F(t)$ (which can trivially be written as $F(t) = \int_{-\infty}^{\infty} \delta(t-t')F(t') \, dt'$) is

$$q(t) = \int_{-\infty}^{t} G(t, t')F(t') \, dt' = \int_{-\infty}^{t} \sin(t-t')F(t') \, dt'. \quad (10)$$

**Oscillatory forcing (switched on at $t = 0$):**

$$F(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ F_\omega \cos \omega t & \text{for } t > 0 \end{cases} \quad (11)$$

Please keep in mind that here, $\omega$ can be arbitrary, and it may or may not be anywhere near the natural frequency $\omega_0$. In terms of our scaled time, we must remember that $\omega$ is scaled with $\omega_0$; so when we write $\cos \omega t$ in scaled time, it can instead be written, in terms of the unscaled time, as $\cos(\omega/\omega_0)\omega_0 t$. I hope that makes sense to you! The solution for scaled time $t > 0$ is

$$q(t) = \frac{F_\omega}{1 - \omega^2} (\cos \omega t - \cos t). \quad (12)$$

(The first term oscillating at the drive frequency is the particular integral, the second term oscillating at the natural frequency is the complementary function, with the amplitude and phase obtained from the initial conditions.)

Driving on resonance $\omega = 1$, this expression just gives infinity. We can instead use the Green’s function to calculate the response to a $\cos t$ forcing switched on at $t = 0$

$$q(t) = \int_{0}^{t} \sin(t-t') \cos t' \, dt'. \quad (13)$$
The result is
\[ q(t) = \frac{1}{2} t \sin t \]  
showing oscillations with an amplitude growing linearly in time. Such a growth is called *secular*. Secular growth of oscillations for an undamped system driven at resonance is an important physical process, and is responsible for many interesting phenomena such as the rings of Saturn. We will also find it to be important when we investigate perturbation methods in mechanics next term. But usually it is unphysical. In any case, if it continues indefinitely, it will drive the system out of the small displacement approximation, and non-linear effects will become important. Damping to the rescue!

**General force** \(F(t)\): Once we work out the response of a SHO (with natural frequency \(\omega_0\)) to a sinusoidal driving force \(F(\omega)\) (where here again, \(\omega\) is scaled in units of \(\omega_0\)), we can Fourier decompose an arbitrary driving force as a (continuous or discrete) linear superposition of sinusoidal ones:

\[
F(t) = \Re \left[ \int F(\omega) e^{i\omega t} d\omega \right], \quad \text{with} \quad F(\omega) = |F(\omega)| e^{i\phi(\omega)} \Rightarrow \quad F(t) = \int |F(\omega)| \cos(\omega t + \phi(\omega)) d\omega
\]  

That is, \(F(\omega)\) is, in general, complex. Think of it as the coefficient (complex amplitude) of the component of \(F(t)\) that is oscillating at frequency \(\omega\).

By linearity, if \(F(\omega) \exp(i\omega t)\) drives the oscillator, the inhomogeneous solution must be of the form \(x(t) = x(\omega) \exp(i\omega t)\); and \(x(\omega)\) must be proportional to \(F(\omega)\).

Also by linearity / superposition, if \(F(t) = \int F(\omega) \exp(i\omega t) d\omega\), the inhomogeneous solution must be of the form \(x(t) = \int x(\omega) \exp(i\omega t) d\omega\).

So, if we can solve the problem for any one arbitrary value of \(\omega\), we can use linearity and superposition to get the solution for any time-varying force. Of course, this won’t work for non-linear equations of motion (e.g., the real “grandfather clock” pendulum).

**Damped driven simple harmonic oscillator**

In the real world, mechanical systems usually have some dissipation or damping. Here we generalize the results discussed above for no dissipation to the case where there is dissipation. A common form (not valid for solid on solid friction) is a force proportional to the velocity.

**Undriven motion**

For no drive this leads to the equation of motion
\[
\ddot{q} + \frac{1}{Q} \dot{q} + q = 0
\]  
with \(Q^{-1}\) proportional to the dissipation coefficient. \(Q\) is the *quality factor* of the oscillator: large \(Q\) is high quality (small dissipation); small \(Q\) is low quality (high dissipation).

Sometimes, this type of dissipation force proportional to velocity is shoe-horned into a Lagrangian type approach by introducing a dissipation function \(D(\{\dot{q}_k\})\) such that the equations of motion are
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} + \frac{\partial D}{\partial \dot{q}_k} = 0.
\]
For the damped simple harmonic oscillator the dissipation function (using the scaled time) would be
\[ D = \frac{1}{2} Q^{-1} \dot{q}^2. \]  
(19)

Note that \(2D\) is the rate of energy dissipation.

The general solution to Eq. (17) is
\[ q(t) = A_+ e^{\lambda_+ t} + A_- e^{\lambda_- t} \]  
(20)

with \(\lambda_\pm\) the roots of the polynomial \(\lambda^2 + Q^{-1} \lambda + 1 = 0\)
\[ \lambda_\pm = -\frac{1}{2Q} \pm \sqrt{\frac{1}{4Q^2} - 1}. \]  
(21)

The behavior depends on the sign inside the \(\sqrt{}\).

- \(Q > \frac{1}{2}\) Under-damped \(q(t) = Ae^{-\frac{1}{2Q} t} \cos(\omega' t + \phi)\) with \(\omega' = \sqrt{1 - \frac{1}{4Q^2}}\)
- \(Q < \frac{1}{2}\) Over-damped \(q(t) = A_+ e^{\lambda_+ t} + A_- e^{\lambda_- t}\) with \(\lambda_+, \lambda_-\) both real and negative
- \(Q = \frac{1}{2}\) Critically damped \(q(t) = (A + Bt) e^{-t}\) (special case, solution not pure exponential)

Here is a plot of these solutions (we choose initial conditions \(q(0) = 1, \dot{q}(0) = 0\)).

![Plot of solutions](image)

Figure 1: Time domain response of a damped SHO, varying the damping (ie, the quality factor \(Q\)), keeping the resonant frequency \(f_0 = \omega_0 / 2\pi\) fixed, and with initial conditions \(q(0) = 1, \dot{q}(0) = 0\).

Your car, with its wheels suspended on shock absorbers (which work like velocity dampers), can be crudely modeled as a damped SHO. Imagine that you hit a bump in the road, causing your wheels to be displaced from the main car body. From the figure above, it should be clear that you’ll recover most gently and quickly from the bump if the \(Q\) of the system is close to critical damping.

Once again, remember that we are using scaled dimensionless time. The advantage of this approach is that all damped SHOs are reduced to the study of a single, unitless mathematical equation, with one parameter \(Q\) characterizing the different types of behavior (overdamped, underdamped etc.). Note that the scaled time represents \(\omega_0 t\) in the original units, with \(\omega_0\) the “linear resonance frequency” of the oscillator, and a scaled frequency represents \(\omega / \omega_0\) in the original units.
The energy $E = \frac{1}{2}(q^2 + \dot{q}^2)$ decays as
\[ \dot{E} = \dot{q}(\ddot{q} + q) = -\frac{1}{Q}q^2 \] (22)
where the second equality comes from the equation of motion. Note that the right hand side is always negative (or zero). For large $Q$ oscillatory motion, the energy decays little in one cycle, and averaging over one cycle gives
\[ \langle V \rangle = \langle \frac{1}{2}q^2 \rangle \simeq \langle T \rangle = \langle \frac{1}{2}\dot{q}^2 \rangle \simeq \frac{1}{2}\langle E \rangle \] (23)
and so the average energy decays as
\[ \langle \dot{E} \rangle \simeq -\frac{\langle E \rangle}{Q}. \] (24)
This is exponential decay with a $1/e$ time of $Q$ (dimensionless, because we have scaled time to units of $\omega_0^{-1}$). There are $Q/2\pi$ oscillations in this decay time - a result independent of the time scaling. Another way of saying this is the the fractional energy loss per period (cycle) is $2\pi/Q$.

These expressions are for large $Q$.

**Driven motion**

The *driven damped linear harmonic oscillator* is described by the equation (after scaling out constants)
\[ \ddot{q} + \frac{1}{Q}\dot{q} + q = F(t). \] (25)
Using the same method as before gives the following solutions (for $q(t \leq 0) = 0$):

**Unit step function:** $F(t) = \Theta(t)$ with $\Theta(t)$ the Heaviside function
\[ \Theta(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ 1 & \text{for } t > 0 \end{cases} \] (26)
The solution for $q(0) = \dot{q}(0) = 0$ and for the underdamped case $Q > \frac{1}{2}$, is:
\[ q = 1 - e^{-t/2Q} (\cos \omega' t + \frac{1}{2\omega' Q} \sin \omega' t), \] (27)
with $\omega' = \sqrt{1 - \frac{1}{4Q^2}}$. Check that $q(0) = \dot{q}(0) = 0$, as required to match to $q(t < 0)$.

**Unit impulse:** For a unit impulse at the origin $F(t) = \delta(t)$ the solution is
\[ q(t) = \frac{1}{\omega'} e^{-t/2Q} \sin \omega' t \] (28)
The solution for a unit impulse at time $t'$ (i.e., $F(t) = \delta(t - t')$) gives the Green’s function
\[ G(t, t') = \begin{cases} \frac{1}{\omega'} e^{-(t-t')/2Q} \sin \omega' (t - t') & \text{for } t > t' \\ 0 & \text{for } t < t' \end{cases} \] (29)
assuming no response before the impulse is applied (the casual Green’s function).

Using superposition the solution to a general forcing $F(t)$ is
\[ q(t) = \int_{-\infty}^{\infty} G(t, t')F(t')dt' = \int_{-\infty}^{t} \frac{1}{\omega'} e^{-(t-t')/2Q} \sin \omega' (t - t') F(t')dt'. \] (30)
Oscillatory forcing: We are usually mainly interested in the steady state solution. Write the forcing $F = \cos \omega t$ as $F = \text{Re}[e^{i\omega t}]$, calculate the response $q_c(t)$ to the forcing $F_c(t) = e^{i\omega t}$, and take the real part at the end. For $F_c(t)$ the steady state solution is

$$q_c(t) = A_c e^{i\omega t} \quad \text{with} \quad A_c = A e^{i\phi}, \quad A, \phi \text{ real} \quad (31)$$

and

$$A_c = \frac{1}{(-\omega^2 + 1) + \frac{i\omega}{Q}}. \quad (32)$$

This gives the solution for $F(t) = \cos \omega t$ as $q(t) = A \cos(\omega t + \phi)$. The amplitude is given by

$$A^2 = \frac{1}{(\omega^2 - 1)^2 + \omega^2/Q^2} \simeq \frac{1}{4 (\omega - 1)^2 + 1/(4Q^2)} \quad (33)$$

where the second expression (a Lorentzian) is a good approximation for large $Q$ and near the large response $\omega \simeq 1$. (It is not a valid approximation for $\omega \approx 0$!) For large $Q$ the response is a sharp resonance peak centered at $\omega_r \simeq 1$, with full width at half intensity (half maximum in $A^2$) FWHM $\equiv \Gamma = 1/Q$. Note for large $Q$ the large enhancement at resonance compared with the static response

$$A_{\text{max}} = A(\omega_r) \simeq Q A(\omega = 0). \quad (34)$$

The phase is

$$\phi = -\tan^{-1} \left( \frac{\omega/Q}{1 - \omega^2} \right) \quad (35)$$

and varies from slightly negative for low frequencies, passing rapidly (for large $Q$) through $-\pi/2$ on resonance, and decreasing to $-\pi$ for large frequencies. You should plot $A(\omega), \phi(\omega)$ for various $Q$ to become familiar with the shapes.

This resonant response – an enhanced amplitude or rapidly varying phase over a narrow frequency range – is very widely used in experimental science and technology to extract a desired signal from the background. It is a central operating principle of spectrometers; it is the “Lorentzian lineshape” of atomic spectral lines; it is the “Breit-Wigner” resonant curve in nuclear and particle physics; it is the mechanical response of a system near resonance; it shows up in a zillion places in LIGO, for example. Actually, I’ll go so far as to assert that this is one of the, if not the most important equation and figure in all of physics!

Remember in these expression we are measuring time in units of $\omega_0^{-1}$ with $\omega_0$ the frequency of the undamped oscillator: you should make sure you can translate all these results back into original, unscaled time and frequency units.

Figure 2 is an example of the behavior of $|A^2|$ and the phase $\phi$ of $A(\omega) = |A(\omega)| \exp i\phi(\omega)$:

Note that as $Q \to \infty$, the amplitude response is a $\delta$ function; infinite at the resonant frequency $\omega = 1$ (or in unscaled units, $\omega = \omega_0$) and zero at all other frequencies. This corresponds to an (unphysically) perfect bell. The impulse response is an undamped sinusoid at the resonant frequency. For finite $Q$, the response is a damped sinusoid, exponential decay (ringdown) with a $1/e$ scaled time of $Q$ (or unscaled time $Q/\omega_0$). The Lorentzian in frequency space is the Fourier transform of a ringdown (exponentially decaying sinusoid) in time.
Return to mass and spring

Let’s return to our mass-and-spring example, with an oscillatory driving force \( F(t) = F_0 \exp(i\omega t) \)

\[
F = m\ddot{x} = -kx - b\dot{x} + F_0 e^{i\omega t}. \tag{36}
\]

By linearity, the particular solution to this inhomogeneous differential equation must be of the form \( x = x_0 \exp(i\omega t) \). Plugging in, and canceling the \( \exp(i\omega t) \) in every term, we get

\[
F = -m\omega^2 x_0 = -m\omega_0^2 x_0 - i\omega b x_0 + F_0, \tag{37}
\]

\[
\Rightarrow x_0 = \frac{F_0}{m\omega_0^2 - m\omega^2 + i\omega b} = \frac{F_0/(m\omega_0^2)}{1 - \omega^2/\omega_0^2 + i\omega/(\omega_0 Q)}, \tag{38}
\]

(see Eqn [32], where \( Q \equiv m\omega_0/b \) is the quality factor discussed above. Low damping (small \( b \)) means high quality factor, strongly peaked Lorentzian amplitude response.

The response to an oscillatory forcing we have calculated can be used to calculate the response to an arbitrary \( F(t) \) using the method of Fourier transforms:

\[
F(t) = \int \tilde{F}(\omega) e^{i\omega t} \Rightarrow x(t) = \int \tilde{x}(\omega) e^{i\omega t} \tag{39}
\]

where linearity guarantees that for each value of \( \omega \), \( x(\omega) \) is the solution to the inhomogeneous equation with sinusoidal driving force proportional to \( F_0 = F(\omega) \). This linearity means that the motion in the frequency domain \( \tilde{x}(\omega) \) is in response only to the force at that frequency, and at no other frequency \( \omega_2 \neq \omega_1 \); the SHO modes are decoupled. This is, in fact, the most important reason why we analyze normal modes in the frequency domain.

One must also add the solution to the homogeneous equation, \( x_0 \exp(i\omega_0 t) \).
The loss angle

One way of incorporating dissipation, in the neighborhood of the resonant frequency, is to endow the spring constant with an imaginary part: \( k \rightarrow ke^{i\phi} \rightarrow k(1 + i\phi) \) (with the last substitution true if \( \phi \ll 1 \) radian). In this context, \( \phi \) is a (usually very small) loss angle. We then have \( \omega_0^2 = k/m \rightarrow \omega_0^2(1 + i\phi) \), and

\[
F = -m\omega^2x_0 = -m\omega_0^2x_0 - im\omega_0^2\phi x_0 + F_0,
\]

(40)

where now, \( \omega_b \) is replaced by \( m\omega_0^2\phi \). The dependence on \( \omega \) is now different, but in the vicinity of resonance, \( \omega \approx \omega_0 \), and it’s not so different (both forms are approximations to the true damping physics, which in general is pretty complex). Now we have \( Q \equiv m\omega_0/b = 1/\phi \), so the loss angle (in radians) is the inverse of the quality factor. Low loss (dissipation) means high \( Q \) (a good bell).

Of course, spring constants are real, not complex, quantities. But as in the case of complex exponentials, it is a mathematical convenience that captures a lot of the physics. The real part of the spring constant is the reactive part, responding in phase with the acceleration, while the imaginary part is the dissipative part, responding 90° out of phase with the acceleration (recall, \( i = \exp i\pi/2 \)); this will give “viscous” damping to the motion.