Physics 106a, Caltech — 17 October, 2019

Lecture 6: Equilibria and Oscillations

We study *equilibria* and *linear oscillations*. Motion of dynamical systems can be classified as *secular* (long-term, non-periodic) or *periodic* (returning, after one period, to its starting point). The most common examples of periodic motion are *orbital motion* (to be discussed next week), and *oscillations* about equilibrium positions. This last type of motion is the subject of today's lecture. You've probably seen most of this material before, in Ph1 and Ph12a.

Example systems of simple harmonic oscillators (SHO)

- Mass on a spring: including a driving force, and friction proportional to velocity (eg, from a dashpot).
- Simple pendulum, in the linear (small angle) regime: include a driving force, and some (velocity-dependent) dissipation at the flexing of the suspension point, or due to air friction.
- Series LRC circuit, with a driving voltage, and an inductance L providing dissipation.
- Elastic bulk material, with $3M$ degrees of freedom (M is of order 10^{24}). After a normal mode (collective motion) analysis (Lecture 18, this term), we consider only the longest wavelength, lowest frequency normal modes of oscillation. The longest wavelength mode is equal to twice the longest dimension of the block. All the higher frequency modes will be observed only as dissipation into heat (Debye heat capacity).
- Elastic rod, with applied linear force: $F = -Y(\Delta L/L)$. Here, F is the *stress*, $(\Delta L/L)$ is the (unitless) *strain*, and Y is the *Young's modulus*, which quantifies the elasticity of the material. The Young's modulus is an *intensive* quantity; it is only a property of the material, and doesn't scale with the mass or size of the object. No real object is truly elastic; there is always some *dissipation* of energy into higher frequency modes (heat).
- Elastic body, with applied shear force (requiring 2 dimensions): $F/A = -G(\Delta L_y/L_x)$. This is a 2-dimensional version of the stress-strain relationship. G is the *shear modulus*; like the Young's modulus, it is an intrinsic quantity.
- Elastic body, with applied pressure force (requiring 3 dimensions): $\Delta P = -\kappa(\Delta V/V)$. Here, the stress is the change in pressure, the strain is the fractional change in the volume, and the elastic constant of proportionality is κ, the *bulk modulus*; again, an intensive quantity.
- Small oscillations about some equilibrium position of a macroscopic system, such as a buoyant object floating in a liquid.
- Small perturbations on a stable orbit, such as planets around the sun, or a particle beam in an accelerator, will cause the particles to execute oscillations about the stable orbit.

SHOs are everywhere

Lest you think that all SHOs are really very simple, let me list some of the ways that they lie at the heart of, say, LIGO and gravitational wave science:

- Gravitational waves are waves, which we routinely Fourier decompose into sinusoids. Each sinusoid is a SHO.
- Environmental noise: The LIGO detectors live on the earth, which has a noisy environment; everything is vibrating. Again, we routinely Fourier decompose the environmental noise into sinusoids (noise spectra). Each component of the spectrum is a SHO.
- Seismic isolation: we isolate the mirrors in our laser interferometer from vertical seismic motion using masses and springs.
- Pendulum suspensions: we further isolate the mirrors, in the horizontal dimension (along the laser beam) by suspending them on a pendulum, which is itself suspended on a pendulum, and again and again. We use *quadruple pendulum suspensions* in advanced LIGO; these are complicated devices!
- Laser light: Even at the classical level, laser light is a collection of modes of oscillation of the electromagnetic field (an ideal laser operates at a single frequency, or a single mode). Each mode is a SHO. At the quantum level, in quantum electrodynamics (QED), each of these modes is a quantum harmonic oscillator.
- Optical cavities: we form resonant optical cavities by pointing two suspended mirrors at each other (they're 4 km apart, in a vacuum system). The laser light resonates in these cavities in (ideally) one normal mode of oscillation, the lowest-order *eigenmode* of the cavity (such as TEM_{00}).
- Quantum noise: Laser light is quantum, not classical. You already know from Ph 12b that we often analyze quantum systems in the Fourier domain; as a wave theory, this is natural. Each mode of the electromagnetic field is a quantum SHO (QSHO), with a quantum zero-point energy and a Heisenberg uncertainty; this is quantum noise, and it ultimately limits the sensitivity of the LIGO detectors. But, we have ways to manipulate the quantum noise of a QSHO, without evading Heisenberg uncertainty, using quantum squeezing techniques.
- Feedback servos and controls: LIGO is a very elaborate control system! We engineer the control plant to be *linear* and *time-invariant* (LTI). Although feedback filters typically work in the *time domain*, with very strong coupling of the system between time steps, its behavior is more effectively analyzed in the *frequency domain*, where it appears as a (continuum of) independent (uncoupled) SHOs.
- Electronics: LIGO has a huge analog and digital electronics plant. All of it (especially the analog electronics) is understood as a (continuum of) independent SHOs.
- Astrophysical sources: Objects like black holes and neutron stars don't emit gravitational waves unless they are perturbed; maybe they are in binaries, and the two stars are in the process of smashing into each other. This causes the stars to get highly perturbed, ringing up their natural, normal modes of oscillation; each *quasi-normal mode (QNM)* is a damped SHO. Even an isolated star (such as a neutron star) can be excited through *star-quakes*. IN all these cases, they release this energy of perturbation in gravitational-wave ringdowns. Studying the modes of a perturbed neutron star using gravitational waves allows us to probe the nuclear force in the most dense and hot conditions imaginable. Studying the modes of a perturbed black hole allows us to study gravity and test general relativity in the most strong-field, highly-dynamical regime imaginable.

Example: damped, driven mass-on-spring

The EOM, including Newton's second law, Hooke's Law, velocity damping, and some external driving force, is:

$$F = m\ddot{x} = -kx - b\dot{x} + F_x^{ext}(t), \tag{1}$$

where m is the mass, k is the spring constant, b is the velocity ($\dot{x}$) damping constant, and F_x^{ext} is the arbitrary, externally applied force in the x direction.

If $F_x^{ext} = 0$ and $b = 0$, we have a linear, homogenous equation $m\ddot{x} = -kx$, with solutions $x(t) = x_c \cos\omega_0 t + x_s \sin\omega_0 t$. Here, $\omega_0 = \sqrt{k/m}$ is the natural frequency of the (single) mode of oscillation. x_c and x_s are constants that can take on any value. Because this is a 2nd order EOM, the solution requires two $t = 0$ initial conditions, $x(0)$ and $\dot{x}(0)$; these can be easily related to x_c and x_s.

If F_x^{ext} = a constant, eg, $-mg$, it will stretch the spring, moving the mass to a new equilibrium point, and the mass will execute SHM about that new equilibrium point. It is often convenient, in that case, to redefine the "zero" of x to be at that equilibrium point.

We can keep both the cos-like and sin-like solutions by writing

$$x = \mathrm{Re}(x_0 e^{i\omega_0 t}). \tag{2}$$

The real part is the cos-like solution; the imaginary part is the sin-like solution. The complex amplitude x_0 describes both the magnitude and the phase of the oscillations:

$$x(t) = \mathrm{Re}(x_0 e^{i\omega_0 t}) = \mathrm{Re}(|x_0| e^{i\phi_0} e^{i\omega_0 t}) = |x_0| \cos(\omega_0 t + \phi_0). \tag{3}$$

Here again, the two $t = 0$ initial conditions will fix the two free parameters, $|x_0|$ and ϕ_0. Note that in this language, the velocity $\dot{x}(t)$ is 90° out of phase with respect to the position.

The full complex exponential is *not* a solution: the displacement $x(t)$ is a real, not complex, quantity. Using the complex exponential form of the solution avoids tedious trigonometry; but in the end, you *must* take the real (or imaginary) part at the end of the calculation. In particular, you must remember to form the real physical solution *before* calculating nonlinear quantities such as x^2.

For this un-damped, un-driven case, the kinetic energy is

$$T = \frac{1}{2} m\dot{x}^2 = \frac{1}{2} m\omega_0^2 |x_0|^2 \sin^2\omega_0 t. \tag{4}$$

and the potential energy is

$$V = \frac{1}{2} kx^2 = \frac{1}{2} m\omega_0^2 |x_0|^2 \cos^2\omega_0 t. \tag{5}$$

Note that the total energy is

$$E = T + V = \frac{1}{2} m\omega_0^2 |x_0|^2, \tag{6}$$

independent of time. The energy oscillates between kinetic and potential.

If we average over one period, using $< \sin^2\omega_0 t > = < \cos^2\omega_0 t > = 1/2$, we get

$$< T > = \frac{1}{4} m\omega_0^2 |x_0|^2; \quad < V > = \frac{1}{4} m\omega_0^2 |x_0|^2, \tag{7}$$

so that $< T > = < V >$ and $E = 2 < T > = 2 < V >$, which is the *Virial Theorem* for a SHO (*not* to be confused with the Virial Theorem for planetary motion, which is completely different, and which we will discuss next week.)

We will return to this example at the end of this lecture.

Lagrangians in the presence of equilibria

Back to a more general or formal notation.

In Lecture 2 of this class, we employed general space-time symmetries (homogeneity, isotropy, and Galilean invariance) to argue that the Lagrangian for a free particle must take the simple form $L = \frac{1}{2}mv^2$ where $v = \dot{q}$. The presence of a conservative force (which breaks homogeneity, and also isotropy unless it's a *central* force) so that $\vec{F} = -\vec{\nabla}V$ must result in a Lagrangian of the form $L = T - V$, where V is a function of coordinates only, not velocities. In that more general case, the kinetic energy can also depend on coordinates, eg, $L = \frac{1}{2}M(q)\dot{q}^2$, but must be quadratic in the velocities.

Here, instead of building in space-time symmetries, we will build in the existence of an equilibrium point. If there exists an equilibrium point q_{eq} (which we are free to set to be zero, $q_{eq} = 0$), than a particle at rest ($\dot{q} = 0$) at that point will stay there: $\ddot{q} = 0$.

What is the most general Lagrangian consistent with the existence of such an equilibrium point? Let's consider the case with one degree of freedom, $k = 1$, so that we don't have to keep writing q_k's; but the problem generalizes to $k > 1$ trivially.

To write the most general Lagrangian $L[q, \dot{q}, t]$, let's Taylor-expand it:

$$L[q, \dot{q}, t] = A + Bq + C\dot{q} + Dq^2 + Eq\dot{q} + F\dot{q}^2 + \text{terms of higher power in } q,\ \dot{q} + G(t)\,, \qquad (8)$$

where $A, B, ...$ are constants, independent of q, $\dot{q}$; $G(t)$ is not constant but it is also independent of q, $\dot{q}$. Of course, A, etc, might depend on powers of q and $\dot{q}$, but we'll absorb that into the "higher powers" terms.

We show below that the existence of an equilibrium point means that we can set $A = 0$, $B = 0$, $C = 0$ and $E = 0$, and since a time-varying external potential will typically ruin an equilibrium point, we'll also set $G(t) = 0$. You can see this by inserting the above Lagrangian into the Euler-Lagrange equation, and look for an equilibrium solution of the form $q(t) = 0$, $\dot{q} = 0$, and $\ddot{q} = 0$.

This leaves the most general Lagrangian consistent with the existence of an equilibrium point to be of the form

$$L = Dq^2 + F\dot{q}^2 + \text{terms of higher power in } q,\ \dot{q}\,. \qquad (9)$$

If you compare this form with the Newtonian description of Hooke's Law above, you see that they are identical in functional form, with D playing the role of $-1/2k$ (the potential energy in Hooke's law), and F plays the role of $1/2m$ in the kinetic energy; it describes simple harmonic motion. Since $m > 0$ in most physics problems, we will assume $F > 0$.

We therefore can *derive* Hooke's law and SHM by simply positing a Lagrangian that supports the existence of an equilibrium point, and working out the consequences for the general form of the Lagrangian (and the Euler-Lagrange equations that follow). The terms in higher powers of q, $\dot{q}$ introduce *anharmonicities.*

This is a general technique when looking for *new* physics: posit symmetries and constraints (here, the constraint that there must exist an equilibrium point solution), construct the most general Lagrangian consistent with those symmetries and constraints, and work out the consequences for the equations of motion. Compare with experiment; if they don't agree, you need to revisit those assumed symmetries and constraints.

Equilibria

An *equilibrium point* or *fixed point* is a time independent solution in some reasonable coordinate system[1]. If we set the initial condition of coordinates $\{q_k\}$ to the equilibrium point and velocities $\{\dot{q}_k\} = 0$, then we require $\{\ddot{q}_k\} = 0$ for the particle to remain there. In the Euler-Lagrange equation which tells us $\{\ddot{q}_k\}$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \tag{10}$$

expand out the first term in partial derivatives

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}_k}\right) + \frac{\partial}{\partial q_k}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\dot{q}_k + \frac{\partial}{\partial \dot{q}_k}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\ddot{q}_k. \tag{11}$$

At an equilibrium the second and third terms on the right hand side are zero. If we restrict our attention to time independent Lagrangians, the first term on the right hand side

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial}{\partial \dot{q}_k}\left(\frac{\partial L}{\partial t}\right) \tag{12}$$

is also zero. I will make this restriction.[2] Then we see from Eq. (10) that the condition for an equilibrium is

$$\left.\frac{\partial L}{\partial q_k}\right|_{\dot{q}_k=0} = 0, \qquad \text{for Lagrangians with no explicit time dependence.} \tag{13}$$

Solve this for the time independent q_k. For time independent holonomic constraints the kinetic energy T is a quadratic form in $\dot{q}_k$ and so at equilibrium $\partial T/\partial q_k = 0$ because $\dot{q}_k = 0$. In this case the equilibrium condition reduces to

$$\frac{\partial V}{\partial q_k} = 0, \qquad \text{for time independent constraints,} \tag{14}$$

so the potential is then a maximum, minimum, or (in greater than 1 dimension) a saddle.

For the rest of the lecture I simplify to systems with a single degree of freedom q. Let's choose the coordinate so that an equilibrium is at $q = 0$. Expand the Lagrangian for q near the equilibrium (small q). Ignoring terms that do not contribute to the equations of motion (e.g. time differentials of functions of q) gives

$$L = Dq^2 + F\dot{q}^2 + \cdots \tag{15}$$

where a possible term Bq is absent by the equilibrium condition which gives $B = 0$. The coefficient F is an effective mass for a kinetic energy, and so I assume it is positive.

[1] Of course for any dynamic solution $q = q_d(t)$ we can define — once we know the solution — a new coordinate $q - q_d(t)$ that is time independent. So by reasonable, I mean one that is chosen before you know a specific solution. However "time independent" does depend on the choice of coordinate system. We might often take a time independent solution in a rotating frame to be an equilibrium, even though it is then time dependent in a nonrotating frame.

[2] One case of a time dependent Lagrangian leading to equilibria of interest is the one for the dynamics inside a uniformly accelerating rail car (rocket ship, Volkswagen ...). As we saw, this leads to a constant effective force, and so to possible equilibria. For example: A child is holding a balloon in a Volkswagen moving with constant positive acceleration a. What is the angle of the string to the vertical in equilibrium? Is the balloon in front or behind of the child's hand?

Equations of motion, scaled variables, and stability

The Euler-Lagrange equation gives the equation of motion for small q, neglecting higher order terms beyond linear ones,

$$\ddot{q} - (D/F)q = 0. \tag{16}$$

This equation is easily solved. For $D/F > 0$ (L a minimum, or V a maximum – remember F is assumed to be positive) there is a solution that grows exponentially $q \propto e^{\sqrt{D/F}t}$, taking q away from the equilibrium point. This is an *unstable* equilibrium. For $D/F < 0$ (L a maximum or V a minimum) the solutions are oscillatory with frequency $\omega_0 = \sqrt{|D|/F}$, and q remains near the equilibrium point if the initial $q, \dot{q}$ are small.

We can eliminate the constants D, F from the analysis by introducing a scaled time variable $\bar{t} = \sqrt{|D|/F}t$ to give

$$\frac{d^2q}{d\bar{t}^2} \pm q = 0, \tag{17}$$

with the plus sign corresponding to $D < 0$ (stable) and the minus sign to $D > 0$ (unstable). The solutions are

$$q(t) = \Re(q_0 e^{\pm it}) \text{ or } \Re(q_0 e^{\pm t}). \tag{18}$$

This rescaling of time means that we need to mathematically investigate just *two equations* (one for the stable case, one for the unstable) to study the behavior near an equilibrium for *any* one dimensional system. Once we have found the mathematical solution (it will be a function of the scaled time $\bar{t}$, of course) we can return to physical units by setting $\bar{t} \to \omega_0 t$ with $\omega_0 = \sqrt{|D|/F}$. Rescaling variables to a dimensionless form to eliminate redundant constants introduced by our choice of units is a powerful technique for collapsing many physical problems onto a reduced family of mathematical ones. This is exploited a great deal in fluid mechanics, where there are often many parameters characterizing the physical system; hence the terms you will hear in this field such as *Reynolds number*, *Rayleigh number* etc. Once the rescaling is done, the bar over the new variable is often dropped to save writing, and we have to remember that we are using the scaled time variable. The technique is not really necessary for the simple equation (16) and some students prefer to keep the original equation, perhaps writing $\sqrt{|D|/F} \to \omega_0$, to immediately see how this parameter affects the results derived. But you should remember it for more complicated problems, and I will use it here.

Using the scaled units, we can make a table of the two types of behavior

Stability	Lagrangian	EOM	Solutions	Hamiltonian	Potential
Stable	$\frac{1}{2}(\dot{q}^2 - q^2)$	$\ddot{q} + q = 0$	$e^{\pm it}$; $\cos t, \sin t$	$\frac{1}{2}(\dot{q}^2 + q^2)$	Minimum
Unstable	$\frac{1}{2}(\dot{q}^2 + q^2)$	$\ddot{q} - q = 0$	$e^{\pm t}$	$\frac{1}{2}(\dot{q}^2 - q^2)$	Maximum

In the unstable case, q grows and eventually reaching large values outside the the range of validity of the small q expansion, and little more can be said with any generality. For the stable case, a small initial condition leads to $q(t)$ remaining small, and so we can study the general behavior of this important case further — linear oscillations near a stable equilibrium.

Undamped simple harmonic oscillator (SHO)

Equation of motion

The equation of motion is

$$\ddot{q} + |D/F| q = 0. \tag{19}$$

Solution

Again, we will implicitly redefine the time variable from t to $\tau = \omega_0 t$, with $\omega_0 = \sqrt{|D|/F}$. The "dot" notation, in $\dot{q} = dq/dt$ will be redefined to mean $\dot{q} = dq/d\tau$. If that isn't enough to confuse you, we will also drop τ an simply use t. So, whenever you see t in what follows, think $\tau = \omega_0 t$, the scaled time.

Get used to using scaled variables, they are everywhere in physics! For example, particle physicists set $c = \hbar = 1$ and measure everything in units of energy, GeV. Relativists set $G = c = 1$ and measure everything in units of cm, or solar mass, or dimensionless units.

With scaled time, the equation of motion is

$$\ddot{q} + q = 0\,. \tag{20}$$

This is a 2nd order (two time derivatives), linear (single power of $q, \dot{q}$ etc.), homogeneous (no constant or $f(t)$ term) ODE (ordinary differential equation) with constant coefficients. For linear, homogeneous equations the principle of the superposition applies: if $q_1(t)$ and $q_2(t)$ are solutions then $q = aq_1(t) + bq_2(t)$ is also a solution. A second order ODE solved as an initial value problem (solution defined by conditions at some initial time we set to $t = 0$) requires two initial conditions $q(0)$ and $\dot{q}(0)$, as we expect for a mechanics problem. We can write the general solution as the superposition form where $q_1(t), q_2(t)$ are *any* choice of two linearly independent solutions, and the constant a and b are fixed by the initial conditions.

The constant coefficients mean that an exponential satisfies the equation: $q \propto e^{\lambda t}$ where, substituting in Eq. (20)

$$\lambda^2 + 1 = 0 \qquad \text{giving} \qquad \lambda = \pm i\ . \tag{21}$$

The general solution is therefore

$$q(t) = ae^{it} + be^{-it}. \tag{22}$$

The (complex) constants a and b are fixed by the initial conditions $a = [q(0) - i\dot{q}(0)]/2$, $b = [q(0) + i\dot{q}(0)]/2$, so that the solution becomes

$$q(t) = q(0)\frac{1}{2}(e^{it} + e^{-it}) + \dot{q}(0)\frac{1}{2i}(e^{it} - e^{-it}) = q(0)\cos t + \dot{q}(0)\sin t. \tag{23}$$

Since $q(0)$ and $\dot{q}(0)$ are real, this shows that the solution is real, as must be true physically, although we are used complex notation to arrive at the result. Of course the combination of exponentials are just $\cos t$ and $\sin t$ and another way to solve Eq. (20) is to remember that a sinusoidal function satisfies this equation, but this approach often leads to more complicated algebra.

An alternative scheme is to note that since $q(t)$ is real, the be^{-it} must be the complex conjugate of ae^{it} so we can write the solution as

$$q = \mathrm{Re}\left[A_c e^{it}\right] \qquad \text{with } A_c = a/2. \tag{24}$$

Writing $A_c = Ae^{i\phi}$ with A the amplitude and ϕ the phase (both real) gives

$$q = A\cos(t + \phi), \tag{25}$$

with A, ϕ set by the initial conditions. You will often see the solution written as Eq. (24) as

$$\text{“ } q(t) = A_c e^{it} \text{ ”} \tag{26}$$

(without the quotes, but they should really be there!) with the understanding that the physical $q(t)$ is given by taking the real part at the end of the calculation. (Hand and Finch write this q_{complex} but then drop the "complex".) This is a *dangerous practice*. For example if we want the quantity $q^2(t)$ such as in the energy, this is *not given* by squaring the complex form $A_c e^{it}$ and then taking the real part: in nonlinear terms you *must* form the correct (physical) function first $\text{Re}[A_c e^{it}]$, and then calculate the nonlinear term (e.g. the square). Check this for yourself.