Physics 106a, Caltech — 8 October, 2019

Lecture 3: Examples of the Lagrangian Approach

Review of the Lagrangian approach

- Use N generalized coordinates that define the configuration at each time

$$q_1(t), q_2(t) \dots q_N(t) \quad \rightarrow \quad \{q_k(t)\}, \; k = 1 \dots N$$

- Make the action $S = \int L \, dt$ (with $L = T - V$) stationary over paths

$$\delta S = \int \sum_k \left[\frac{\partial L}{\partial q_k} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) \right] \delta q_k(t) \, dt = 0$$

- If the N coordinates are independent

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = 0 \qquad \text{for } k = 1 \dots N$$

  and if a change of each coordinate is consistent with any constraints then $L = T - V$ can be evaluated without knowing any constraint forces

- For unconstrained dynamics of M particles $N = 3M$. For constrained dynamics N may be reduced.

- For most problems

  - Use N generalized coordinates $q_1(t), q_2(t) \dots q_N(t)$ that
    * define the configuration at each time
    * build in any symmetries and constraints (eg, isotropy $\rightarrow$ use angles!)
    * can be varied independently, consistent with any constraints

- For each coordinate use the Euler-Lagrange equation

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = 0 \qquad \text{for } k = 1 \dots N$$

  with $L = T - V$.

Recovering Newton's Laws:

For the elementary description of M particles with position vectors $\vec{r}_i$ and Cartesian coordinates (x_i, y_i, z_i), for $i = 1 \dots M$, and including all the forces, assumed conservative, in the potential $V(\{\vec{r}_i\})$

$$L = \sum_{i=1}^{M} \tfrac{1}{2} m_i (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) - V(\{\vec{r}_i\})$$

The Euler-Lagrange equation for x_i is

$$\frac{d}{dt}(m_i \dot{x}_i) + \frac{\partial V}{\partial x_i} = 0$$

which is Newton's 2nd law of motion for conservative forces. So, the Euler-Lagrange equations are equivalent to Newton's 2nd law, in the simple cases where one would simply apply Newton's 2nd law: Cartesian coordinates, no constraints, conservative forces.

General approach

The general approach to a wide class of problems is

- Choose N generalized coordinates q_k that define the configuration and can be varied independently whilst maintaining any constraints. Choose coordinates that best suit the problem; they need not be Cartesian, they need not have units of length, they not be in the same reference frame.
- Evaluate the Lagrangian $L = T - V$ in some inertial frame in terms of these coordinates and their time derivatives. Here T is the kinetic energy and V is the potential energy.
- Solve the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \qquad \text{for } k = 1 \ldots N. \tag{1}$$

For problems that are not naturally expressible in terms of Cartesian coordinates, and especially for problems that have constraints, there will be fewer Euler-Lagrange equations than Newtonian equations; they will be less strongly coupled (in the mathematical sense), and in principle, it will be easier to interpret them.

These results also have general and profound importance, and will be discussed more later.

Some example applications

In class we went through a number of examples demonstrating the use of the Lagrangian:

1. From the calculus of variations: the shortest path between two points in a 2D plane. This is math (x vs y), not physics (dynamics of $x(t)$), so the integral is over a "length functional", not the Lagrangian.
2. Unconstrained motion in 2D, but in polar rather than Cartesian coordinates. The angular equation of motion boils down to $\dot{L} = F_\phi$, where L here is the angular momentum (a scalar, in 2D), and F_ϕ is the *torque*. If the force is *central* (so that $V(\vec{r})$ depends only on $|\vec{r}|$, not the polar angle) then $F_\phi = 0$ and angular momentum is conserved. This generalizes to 3D with polar angle θ and azimuthal angle ϕ, and a central force results in conservation of $\vec{L}$, the angular momentum 3-vector.
3. Plane pendulum in 2D. Naively there are two coordinates, x and y, and constraint $\sqrt{x^2 + y^2} = \ell^2$. Expressed instead in terms of one independent coordinate $\phi(t)$, we have

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\ell^2\dot{\phi}^2; \quad V = mg\ell(1 - \cos\phi); \quad L = T - V. \tag{2}$$

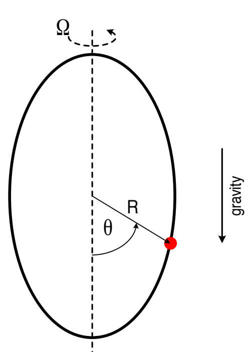
One can then plug this into the Euler-Lagrange equations, resulting in one equation of motion to be solved. Linearizing gives SHO; higher amplitude oscillations give non-linear "anharmonicity".

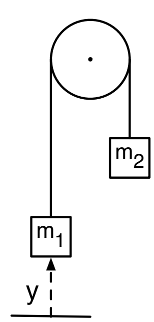
4. Pendulum on a rotating support.
5. The double pendulum - more complicated!
6. The block and wedge problem...
7. Masses on frictionless pulley(s) ("Atwood machine")
8. Motion in an accelerating rail car or spaceship
9. A particle sliding around the interior of a frictionless cone (the rolling ball equivalent is often used in science museums to illustrate planetary orbits)
10. A bead on a rotating stiff frictionless wire (the wire traces out a conical surface)
11. Continuum elastic string (if time)
12. Precessing top - this will be a big subject later in the term!

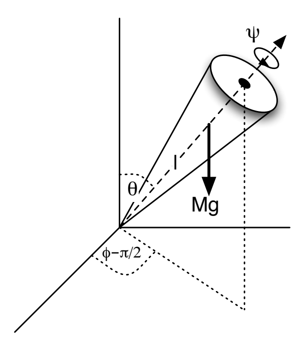
An instructive example is a bead on a circular wire (hoop) of radius R, with gravity g, and spinning with angular velocity ω. Here we again have one generalized coordinate θ, one Euler-Lagrange equation, and thus one equation of motion (EOM). It has four solutions, each with different stability properties: $\theta_1 = 0$ (bottom of the hoop, stable); $\theta_2 = \pi$ (top of the hoop, always unstable); $\theta_{3,4} = \pm\text{arccot}(g/\omega^2 R)$ (stable iff $g < \omega^2 R$).

If the hoop is spinning slowly enough, the bead is stable at the bottom of the hoop. If it is spinning fast enough ($\omega^2 > g/R$), the bead rises up the hoop, either to the left or the right, to θ_3 or θ_4; ie, it *bifurcates* (this will be an important effect when we consider chaotic systems). There's a symmetry between solutions θ_3 and θ_4; the system chooses one direction or another, based on tiny fluctuations of the motion: *spontaneous symmetry breaking.*

Are the solutions stable? Considering a small deviation $\delta\theta$ around a solution, the EOM will be of the form $\ddot{\delta\theta} = c\delta\theta$, where c is some constant that will depend on things like m, g, etc. If the constant $c < 0$, $\delta\theta$ will oscillate stably about 0 (the equilibrium solution), and if $c > 0$, the equilibrium is unstable.







Conjugate momenta, ignorable coordinates, conserved momenta

Quite generally, we define the *conjugate momentum* p_k of the generalized coordinate q_k as

$$p_k \equiv \frac{\partial L}{\partial \dot{q}_k}. \tag{3}$$

If a coordinate q_m does not explicitly appear in the Lagrangian, i.e.,

$$\frac{\partial L}{\partial q_m} = 0 \tag{4}$$

it is called *ignorable* or *cyclic*. Equation (1) immediately gives for this coordinate

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_m}\right) = \dot{p} = 0. \tag{5}$$

Thus the momentum conjugate to an ignorable coordinate is a constant of the motion. In the simplest case, if the Lagrangian $L = T - V$ doesn't depend on a coordinate, so that a change to that coordinate does not change the Lagrangian dynamics (ie, a symmetry), then there's no force along that direction: $F_m = -dV/dq_m = 0$, so the (generalized) momentum in that direction is conserved. This connects *symmetries* to *conserved quantities* (Noether's Theorem). We will discuss this more later.

The Hamiltonian, time independent Lagrangians, conservation of energy

Consider the *total* time derivative of $L = L(\{q_i\}, \{\dot{q}_i\}; t)$.

$$\frac{dL}{dt} = \sum_i \frac{\partial L}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i + \frac{\partial L}{\partial t}. \tag{6}$$

Now use the Euler-Lagrange equations, in the form $\partial L/\partial q_i = \dot{p}_i$:

$$\frac{dL}{dt} = \sum_i (\dot{p}_i\dot{q}_i + p_i\ddot{q}_i) + \frac{\partial L}{\partial t} = \frac{d}{dt}\sum_i (p_i\dot{q}_i) + \frac{\partial L}{\partial t}, \tag{7}$$

$$\Rightarrow \frac{d}{dt}\left(\sum_i (p_i\dot{q}_i) - L\right) = -\frac{\partial L}{\partial t}. \tag{8}$$

Let's change coordinates from $q_k, \dot{q}_k$ to q_k, p_k. Define the *Hamiltonian* as

$$H = \sum_k \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L \equiv \sum_k \dot{q}_k p_k - L\,. \tag{9}$$

From the above (taking care on the difference between d/dt and $\partial/\partial t$) shows

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t}\,, \tag{10}$$

i.e. the *total* time derivative of the Hamiltonian along the actual path of the dynamics given by $\{q_k(t)\}$ is given by the *explicit* time dependence of the Lagrangian function. If there is no explicit time dependence in the Lagrangian (no time-varying external forces), then H is a *constant of the motion.*

The transformation from the Lagrangian $L(q, \dot{q}; t)$ to the Hamiltonian $H = p\dot{q} - L$ (here I am suppressing the k's for clarity but with no loss of generality) is called a *Legendre transformation.* You have seen these before, in thermodynamics. For example, we can transform the energy $U(S, V)$ to the Helmholtz free energy $F(T, V) = U - TS$. In thermodynamics, the temperature $T = (\partial U)/(\partial S)$ (keeping the volume V fixed), is the variable conjugate to the entropy S. The total energy is naturally a function of S, since in an isolated system the energy and entropy are both constant. The free energy is naturally a function of T, not S, since a system that can exchange energy with a thermal reservoir at temperature T is naturally described by T.

Similarly here, the Lagrangian is naturally a function of coordinates q_k and velocities $\dot{q}_k$, while the Hamiltonian is naturally a function of coordinates q_k and conjugate momenta p_k. Later on we will explore the utility of this formulation.

The Hamiltonian is often the energy $T + V$. It is so if both the potential energy is independent of velocities, as is usually needed to be able to define a potential, and if the kinetic energy is a quadratic form in $\dot{q}_k$. The latter condition means $T = \frac{1}{2}\sum_{kl} t_{kl}\dot{q}_k\dot{q}_l$ with t_{kl} possibly depending on coordinates $\{q_j\}$ and time t. Then $\sum_k \dot{q}_k \partial T/\partial \dot{q}_k = 2T$, which gives $H = T + V$ (see Hand and Finch, problem 1-9). These properties of the kinetic energy hold for time independent holonomic constraints (see Lecture 4). In this case, if there is no explicit time dependence in the Lagrangian, energy is conserved.

Here, we have introduced the Hamiltonian specifically for the purpose of showing how the Lagrangian formalism, with no explicit time dependence, results in energy conservation. Later, we will develop the Hamiltonian formulation of classical dynamics.