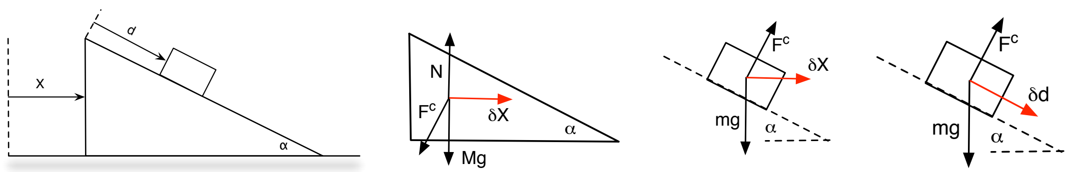
Physics 106a, Caltech — 3 October, 2019

Lecture 2: Variational Approach

Example

We look at a simple example of applying Newton's laws of motion, a block sliding on an inclined plane. This is discussed in Hand and Finch §1.1, and will be used as a recurring example as we proceed. Looks like a simple "toy" high school physics problem...



The leftmost figure sets up the problem, in the 2D space of the page. All surfaces are frictionless. Gravity causes all masses to be accelerated downward with constant acceleration g (except for the table!).

We assume that there's no motion in/out of the plane of the page; but of course that's not true. The center-of-mass of the block and wedge are moving on the frictionless table at some constant velocity $\vec{V}$, constrained to be in the plane of the table. We can apply a Galilean transformation, allowing us to work in the "rest frame" of the block-and-wedge, thereby ignoring the uniform motion on the table. Don't forget about it, or points will be taken off! (Or, if you're an ice hockey player, you will be a very bad one). And, of course, the entire system, including table, is hurtling through space; but we'll make the (hopefully good) approximation that the table is at rest in some inertial frame. (It's not, because the Earth is spinning; as we'll see, we can ignore the resulting *coriolis* force for small objects.)

Now analyze the forces, as in the next two figures. The wedge feels the force of gravity downward, and the force from the table, *normal* to the plane, pushing up with equal but opposite force. The downward gravitational force is a "fundamental" one, easy enough to understand. The normal force from the table is far more complicated, due to the rigidity of the solid table, itself due to the complicated interactions between all of the atoms in the solid; ultimately of electromagnetic (and nuclear) origin; but way too complicated to explain! (There's a famous Feynman story about that). We just intuit that there's no vertical motion, either up (floating) or down (compressing the table), and thus intuit that we must have $N = F_T = Mg + F_c \cos\alpha$ (here, F_T is the force from the table onto the wedge; it is often called the force "normal" to the table, hence the N, but we often use $\vec{N}$ for the torque.)

Note that no table is infinitely rigid. There will be some finite *elasticity*, causing the wedge-and-block to compress the table, increasing the normal force, eventually reaching some quilibrium. Due to the elasticity, there will be some small *simple harmonic motion* about that equilibrium position. Ignoring that, the only "non-trivial" motion is horizontal, along δX. That is one *degree of freedom.*

The block experiences the force of gravity, and the *contraint force* from the wedge, preventing it from falling into the wedge. The net force will cause it to accelerate along X, but the constraint forces it move in both X (horizontal) and Y (vertical).

It clearly would be easier to describe its motion along the wedge's slope, δd, as in the rightmost figure. That motion is easiest to picture relative to the wedge, which is itself accelerating with respect to the inertial frame of the table.

Ugh! It's all worked out, in the Physics 1 Newton's Law way, in H&F section 1.1. They then go on to work it out in terms of work and energy; much simpler.

Important lessons from the example on the practical application of the Newtonian approach are:

1. the "natural" coordinates or *generalized coordinates* $\{q_k\}$ may not be Cartesian coordinates with respect to some inertial frame;
2. we needed to introduce "forces of constraint" that are usually not of interest (they're usually canceled by some other force);
3. we needed to invoke Galilean invariance (a *symmetry* to allow us to ignore many of the other "trivial" degrees of freedom (4 of the 6 - count them!);
4. complicated algebra may be needed to eliminate the "extra" coordinates and forces introduced.

Many of these difficulties are eliminated using the *Lagrangian* approach introduced in this lecture.

Hamilton's Principle

We introduce the *variational approach to classical mechanics*. This is formulated in terms of the *Lagrangian* $L(\{q_k\}, \{\dot{q}_k\}, t)$ which is a function of the particle coordinates $\{q_k\}$ and velocities $\{\dot{q}_k\}$ (which are themselves functions of time), and time t (k runs over the number of coordinates that need to be considered). We're using the symbol q for the coordinate because often we will not use Cartesian coordinates. To emphasize this, we often talk about these as *generalized coordinates*.

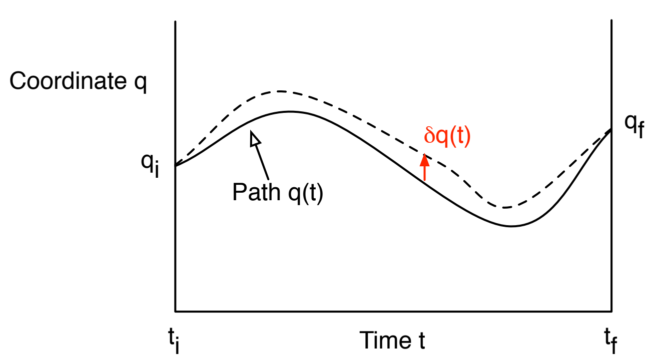
The starting point is *Hamilton's principle*:

The dynamics of the system $\{q_k(t)\}$ from time t_i to time t_f is such that the *action*

$$S = \int_{t_i}^{t_f} L(\{q_k\}, \{\dot{q}_k\}, t)\, dt \tag{1}$$

is *stationary* over all trial paths with fixed endpoints $\{q_k(t_i)\}, \{q_k(t_f)\}$.

Stationary (minimum, maximum, or saddle) means that for an infinitesimal change in path $\delta q_k(t)$ we have $\delta S = 0 + O(\delta q_k^2)$. The following figure for dynamics with a single coordinate may help to explain the procedure. We can also visualize the action S as the integral of the "height" L of each curve (into and out of the plane of the paper). Stationary paths follow ridges or valleys.



Calculus of variations

The *calculus of variations* shows us how to find the stationary value of a *functional* such as the action $S[\{q_k(t)\}]$ with respect to the paths (functions) $q_k(t)$. First consider a single particle coordinate $q(t)$. Consider an infinitesimal change in the path $\delta q(t)$. This gives the change in the Lagrangian at each time by the chain rule

$$\delta L = \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} \tag{2}$$

and so the change in the action

$$\delta S = \int_{t_i}^{t_f} \delta L \, dt = \int_{t_i}^{t_f} \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} \right) \, dt \, . \tag{3}$$

Note that the coordinate $q(t)$ over the path determines the velocity $\dot{q}(t)$, and so $\delta q(t)$ and $\delta \dot{q}(t)$ are not independent variables:

$$\delta \dot{q} = \delta \left(\frac{dq}{dt} \right) = \frac{d}{dt}\delta q. \tag{4}$$

We therefore integrate the second term by parts, using $\delta q = 0$ at the endpoints, to give

$$\delta S = \int_{t_i}^{t_f} \{ \frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \right) \} \delta q(t) \ dt \equiv \int_{t_i}^{t_f} \frac{\delta L}{\delta q}\delta q(t) \ dt, \tag{5}$$

with the *variational derivative* defined as

$$\frac{\delta L}{\delta q} = \frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \right) \, . \tag{6}$$

We also call this the *functional derivative* of S

$$\frac{\delta S[q(t)]}{\delta q} \equiv \frac{\delta L(q, \dot{q}, t)}{\delta q}, \tag{7}$$

(same symbol, different meaning). Hamilton's principle tells us that to get the equations of motion for the classical path, set $\delta S = 0$. Since $\delta q(t)$ is arbitrary we get the *Euler condition* $\delta L / \delta q = 0$, i.e.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0. \tag{8}$$

This gives us the *Euler-Lagrange* equation which is the *equation of motion* (EOM) for the particle. We saw last time that the *work* is the first spatial integral of the EOM $\vec{F} = m\vec{a}$; here we see that the EOM is the first spatial derivative of the Lagrangian, so the Lagrangian is analogous to work or energy.

The generalization to N independent particle coordinates is straightforward, since each arbitrary variation δq_k is independent, and we get N equations of motion $\delta L / \delta q_k = 0$, i.e.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = 0. \tag{9}$$

Note that we are making use of *three* different kinds of small deviations, indicated by different symbols:

- d denotes a *total* derivative, taking into account implicit dependences, for which the chain rule is required;
- ∂ denotes a *partial derivative*; eg, $\partial L / \partial \dot{q}_k$, in which one can assume that q and $\dot{q}$ are independent;
- δ denotes a small variation of a function.

Comments on math

The calculus of variations is used in a much wider class of problems than optimizing the action, for example many problems involve minimizing a number that depends on a function $y(x)$ (x might be space), and derivatives

$$\text{Minimize} \qquad \int_{x_i}^{x_f} F\left(y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, x\right) dx, \tag{10}$$

(the approach generalizes to functions of higher derivatives). Examples are

- Fermat's principle of least time along a light path through lenses and mirrors, in optics;
- the shape of a massive rope stretched loosely between two fixed points, under the influence of gravity (the *catenary*);
- the shape of a frictionless wire in gravity connecting two fixed points that gives the shortest passage of time of a particle released from rest from one end (Brachistochrone problem);
- the shortest distance between two points in flat space (straight line), the surface of a sphere (great circle) or any general (flat or curved) space (geodesic). This is the basis of General Relativity.
- The Feynman path integral approach to quantum mechanics is a generalization of Hamilton's Principle. The requirement $\delta S = 0$ gives the classical path, while deviations $\delta S \sim \hbar$ give possible quantum paths. You should verify that S has the same units as $\hbar$.

See Hand and Finch §2.1-2 and Problems 2-1 to 2-12, and Assignment 1 for a historical discussion and examples. Note that the scheme is set up for path variations with fixed endpoints. Other schemes might sometimes be appropriate, e.g. see Hand and Finch Chapter 2, Appendix A on Maupertuis principle.

Lagrangian for a single particle

Using the idea that the action should reflect the symmetries and invariances of the physical problem, we argue for a specific form for the Lagrangian of a single particle. The Euler-Lagrange equation then turns out to give Newton's first and second laws!

Note that physics is *not* math! There is no rigorous way to *derive* the Lagrangian for a physical system; we are only guided by our intuition, which in turn is guided by our current knowledge of physics (as learned from, eg, Aristotle, Newton, Einstein, ...).

One can be bolder, and less constrained by "the current paradigms", by guessing at new symmetries and interactions; these would be expressed as new terms in the Lagrangian, or new constraints on existing or new terms. These will lead to new laws of physics, testable through careful experimentation. Those who are insightful (or lucky) enough to guess new terms in the Lagrangian of the universe that stand the test of experiment win Nobel prizes. Even if they don't apparently apply to our universe, new Lagrangians can make for very interesting new physics.

Hilbert "derived" Einstein's equations of General Relativity from the action principle with a natural choice of a Lagrangian. Feynman "derived" quantum mechanics from the action principle with a natural choice of a Lagrangian. The Standard Model of particle physics is best expressed this way; so is SuperSymmetry and String theory (both hypothetical theories of nature).

Free space

First consider a particle in free space. I will use Cartesian coordinates, combined into a vector $\vec{r}$ giving the position of the particle. The velocity is then $\vec{v} = \dot{\vec{r}}$. Space is uniform (homogeneity), and so the Lagrangian should not depend on the particle position, only on the velocity. The isotropy of space means that the Lagrangian should only depend on the magnitude of the velocity (the speed), not the its direction, i.e. $L = L(v^2)$. We will build these symmetries into the Lagrangian.

We now use the idea that the equations of motion should be the same in all inertial frames, i.e. invariant under $\vec{v}(t) \to \vec{v}'(t) = \vec{v}(t) + \vec{v}_T$ with $\vec{v}_T$ a constant (the transformation velocity given by the relative motion between the two inertial frames), to restrict the function L. In fact we find that L must be proportional to v^2! To match to the Newtonian formulation, we write the proportionality constant $\frac{1}{2}m$ and call m the mass of the particle. Thus

$$L = \tfrac{1}{2}mv^2, \qquad \text{free particle.} \tag{11}$$

With this Lagrangian, let's see how the action changes under the Galilean transformation. In terms of the new velocity the action is

$$S = \frac{1}{2}m\int_{t_i}^{t_f} (\vec{v}' - \vec{v}_T)^2\,dt = \frac{1}{2}m\int_{t_i}^{t_f} v'^2\,dt - m\vec{v}_T\cdot(\vec{r}_f - \vec{r}_i) + \tfrac{1}{2}mv_T^2(t_f - t_i), \tag{12}$$

where the second term in the last expression comes from integrating $\vec{v}'$ over time, giving the separation between final and end points, which is unchanged by the Galilean transformation. Now we see that the action in terms of $\vec{v}'$ does not have exactly the same form as the original one (i.e. the action is not invariant under the Galilean transformation). However the difference is a *constant* which does not change under path variation (remember the endpoints are fixed), and so the equation of motion are unchanged. You can readily check that this does not work for other functions of v^2, for example v^4. Thus we have derived the Lagrangian and action for a particle in free space, from symmetry arguments.

Note that the Euler-Lagrange equation in this case is $m\vec{v} =$ constant: the uniformity (translational invariance) of space together with Galilean invariance directly leads to momentum conservation and Newton's first law! This is a simple example of the profound relationship between symmetries and conservation laws that we will study in more generality later in the course.

Particle with forces

If the particle is not in free space, we must include a space dependent function to account for this. The simplest choice is just to add a space-dependent function $f(\vec{r})$ to the Lagrangian (rather than make the mass depend on $\vec{r}$ for example). We must then learn about the physics of what produces $f(\vec{r})$, just as Newton's second law tells us to investigate the physics of "forces". It turns out that to relate our new function to Newton's forces, we need $f = -V$, with $V(\vec{r})$ the potential energy, so that

$$L = \tfrac{1}{2}mv^2 - V(\vec{r}), \qquad \text{particle with forces.} \tag{13}$$

Note that we are restricted to *conservative forces*. Many nonconservative forces, such as friction, cannot easily be treated in the Lagrangian formulation. These are not fundamental, microscopic forces, but are derived from the mechanics of a very large number of particles (atoms), which themselves *are* governed by conservative forces. The derivation of such *dissipative* effects is quite subtle, and takes us into the realm of *statistical mechanics* rather than mechanics.

The magnetic force, although nonconservative, *can* be treated in the Lagrangian approach, but via a *velocity dependent* potential: this is described below.

The Euler-Lagrange equation is now easy to write down. (Or, as an exercise, go through the explicit calculus of variations manipulations for this simple example to see how things work out.) We have for the x coordinate, $\partial L/\partial \dot{x} = m\dot{x}$ so the Euler-Lagrange equation is

$$m\ddot{x} + \frac{dV}{dx} = 0, \tag{14}$$

with similar equations for the y, z coordinates. Identifying $-dV/dx$ with force, this is Newton's second law of motion.

General Lagrangian

It is straightforward to generalize the preceding discussion to many particles. This leads us to propose the general form of the Lagrangian for mechanics

$$L = T - V, \tag{15}$$

with T the total kinetic energy and V the total potential energy (including interparticle interactions and interactions with external sources).

Advantages of the Lagrangian formulation

Hamilton's principle (leading to the Lagrangian equation of motion) is an alternative *fundamental statement* of the laws of classical mechanics with a number of advantages over the Newtonian formulation. Some advantages are

- the action is a simple scalar quantity that reflects the symmetries of the physical system, and this is often enough for us to guess the form it must take, as in the single particle problem;
- the principle is easy to derive as the $\hbar \to 0$ limit of quantum mechanics;
- the path variation may be expressed in terms of any convenient coordinates defining the particle positions: we are not restricted to Cartesian coordinates in an inertial frame, or even variables with the dimensions of length;
- for dynamics problems with constraints, such as the sliding block and wedge, we can eliminate the constraints from the formulation by using coordinates whose variations are consistent with the constraints;
- in practice, it is often easy to evaluate the action, since the kinetic energy (scalar function of the scalar speed) is usually easy to evaluate even in complicated situations (unlike the acceleration cf. Coriolis and centrifugal "forces" in noninertial frames — see later);
- the approach generalizes quite naturally to relativistic mechanics, quantum mechanics, statistical mechanics, classical and quantum field theory ....

Most of these ideas will be illustrated in later lectures.

You will explore some examples of the application of Hamilton's principle and the Euler-Lagrange equations in the textbooks and the problem sets; eg,

- the motion of a bead constrained to a circular or elliptical wire;
- motion constrained to the surface of a sphere;
- motion of a cannonball under the influence of gravity;
- the catenary of a hanging rope;
- the oscillations of a mass and massive spring.
- rolling without slipping.

Relativistic Mechanics

It is interesting to try to derive relativistic mechanics using the idea that the action should respect the symmetries of the physics. To derive the equations of relativistic mechanics we can make the hypothesis that the action S is a Lorentz invariant (i.e., unchanged on transforming to a different frame of reference). Since time is *not* the same in different frames, we first write the action of a particle as

$$S = \int \mathcal{L}\, d\tau$$

with τ the *proper time* – the time measured in an inertial frame instantaneously co-moving with the particle – that is Lorentz invariant. Now we look for $\mathcal{L}$, a function of particle velocity and position, that is Lorentz invariant and satisfies other symmetries of the problem. For a free particle, the only possible function is a constant! See if you can show this. Since overall scale factors in the action do not affect the equations of motion we could choose $\mathcal{L} = 1$. To connect with the Newtonian Lagrangian in the small velocity limit we instead use

$$\mathcal{L} = -mc^2,$$

with m the mass of the particle and c the speed of light (both Lorentz invariants!).

To find the equations of motion we work in a particular inertial frame, and then in terms of time t in our frame we have $d\tau = dt/\gamma$ with $\gamma = 1/\sqrt{1 - v^2/c^2}$ (time dilation). Here v is the speed of the particle measured in our frame. Thus, we have for a free particle

$$S = \int L\, dt \quad \text{with} \quad L = -mc^2\sqrt{1 - v^2/c^2}.$$

You can check that in the limit $v \ll c$, this reduces to the Newtonian Lagrangian; this justifies the choice of the constant in the equation above.

Now let's add an electromagnetic field. The only Lorentz invariant we can add to $\mathcal{L}$ that has the right symmetry properties is a constant times $\mathbf{v} \cdot \mathbf{A}$ where $\mathbf{v}$ is the *velocity 4-vector* and $\mathbf{A}$ is the *electromagnetic potential 4-vector* (the dot product of two 4-vector is a Lorentz invariant). Thus for a particle in an electromagnetic field

$$\mathcal{L} = -mc^2 - q\mathbf{v} \cdot \mathbf{A}$$

where again the constant is chosen to match onto Newtonian physics at small velocities: q is then the charge of the particle. In our inertial frame, the components of the 4-vectors are $\mathbf{v} = \gamma(c, \vec{v}), \mathbf{A} = (\Phi/c, \vec{A})$ with Φ the electric scalar potential and $\vec{A}$ the magnetic vector potential. Thus the Lagrangian in our frame of reference is

$$L = -mc^2\sqrt{1 - v^2/c^2} - q\Phi(\vec{x}, t) + q\vec{v} \cdot \vec{A}(\vec{x}, t).$$

With a little effort you can derive the Euler-Lagrange equation, and leading in the Newtonian limit to the equation of motion for a particle with the Lorentz force $\vec{f} = q(\vec{E} + \vec{v} \times \vec{B})$ (see Assignment 1, Problem 4).

Note that this neglects the dynamics of the EM potential itself, eg, due to the "back-action" of the system of charged particles on the (external) source of the field. We will go deeper into relativistic electrodynamics in Ph 106b,c (and in quantum field theory).