Physics 106a, Caltech — 3 December, 2019

Lecture 18 – Relativistic kinematics & dynamics

Momentum and energy

Our discussion of 4-vectors in general led us to the energy-momentum 4-vector $\mathbf{p}$ with components in some inertial frame $(E, \vec{p}) = (m\gamma_u, m\gamma_u \vec{u})$ with $\vec{u}$ the velocity of the particle in that inertial frame and $\gamma_u = (1-u^2)^{-1/2}$.[1] The length-squared of the 4-vector is (since $\mathbf{u}^2 = 1$)

$$\mathbf{p}^2 = m^2 = E^2 - p^2 \tag{1}$$

where the last expression is the evaluation in some inertial frame and $p = |\vec{p}|$. In conventional units this is

$$E^2 = p^2c^2 + m^2c^4 \ . \tag{2}$$

For a light particle (photon) $m = 0$ and so $E = p$ (going back to $c = 1$). Thus the energy-momentum 4-vector for a photon is $E(1, \hat{n})$ with $\hat{n}$ the direction of propagation. This is consistent with the quantum expressions $E = h\nu, \vec{p} = (h/\lambda)\hat{n}$ with ν the frequency and λ the wave length, and $\nu\lambda = 1$ (the speed of light)

We are led to the expression for the relativistic 3-momentum and energy

$$\vec{p} = \frac{m\vec{u}}{\sqrt{1-u^2}}, \quad E = \frac{m}{\sqrt{1-u^2}}. \tag{3}$$

Note that $\vec{u} = \vec{p}/E$. The frame independent statement that the energy-momentum 4-vector is conserved leads to the conservation of this 3-momentum and energy in each inertial frame, with the new implication that mass can be converted to and from energy.

Since $(E, \vec{p})$ form the components of a 4-vector they transform between inertial frames in the same way as $(t, \vec{x})$, i.e. in our standard configuration

$$p'_x = \gamma(p_x - vE) \tag{4}$$
$$p'_y = p_y \tag{5}$$
$$p'_z = p_z \tag{6}$$
$$E' = \gamma(E - vp_x) \tag{7}$$

and the inverse

$$p_x = \gamma(p'_x + vE') \tag{8}$$
$$p_y = p'_y \tag{9}$$
$$p_z = p'_z \tag{10}$$
$$E = \gamma(E' + vp'_x) \tag{11}$$

These reduce to the Galilean expressions for small particle speeds and small transformation speeds. The momentum expression is simply given by setting $\gamma \simeq 1$. For the energy we use $E \simeq m + T$

[1] Remember I'm using bold for 4-vectors and units in which $c = 1$.

with T the Newtonian kinetic energy, and $\gamma \simeq 1 + \frac{1}{2}v^2$. These give to leading order the Galilean expressions

$$p_x = p'_x + mv \tag{12}$$

$$T = T' + vp'_x + \tfrac{1}{2}mv^2 \tag{13}$$

Since these are easy to intuit, they provide a good way of getting the sign right in the p_x and E equations.

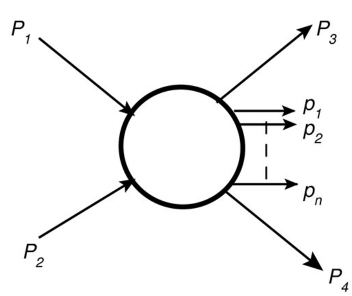
Kinematics, dynamics, collisions

An important topic in relativistic mechanics is collisions, since colliding elementary particles at high energies is the main tool of particle physics. The physics principle is the conservation of 4-momentum.

The motion of "free" particles (experiencing no forces or interactions) are completely described by their 4-vector; this is their *kinematics*. Collisions involve forces or interactions; the *dynamics*. Since in relativity nothing can travel faster than the speed of light, there can be *no action at a distance*; all interactions are *local*, involving the exchange (emission and absorption) of force-carrying particles like the photon.

Since we have seen that relativistic forces are complicated to describe (and rarely used in practical computations), we prefer to use the term *interactions* over *forces*. For example, the Standard Model of particle physics describes three microscopic interactions: strong nuclear (QCD), weak nuclear (QFD), and electromagnetic (QED).

To simplify the picture in micro-physics (quantum mechanics of atoms, nuclei, particles) we break the process up into three parts: an initial state consisting of *asymptotically free* particles approaching each other; a *local interaction* (like a collision) involving changes in the motion; a final state consisting of *asymptotically free* particles flying away from each other. This is an excellent approximation, even in a dense gas where particles might have a *mean free path* between collisions of a micron or less; the interaction still happens in a much smaller region, and the incoming and outgoing particles can still be treated as asymptotically free. Assuming the interaction dynamics conserves energy and momentum, we can use 4-momentum conservation between the initial and final states. Here's a picture of a $2 \to N$ collision process, with time along the horizontal axis:



The incoming (on the left) lines and outgoing (on the right) lines represent particles with momentum 4-vectors; their sums are conserved:

$$\sum_i \mathbf{p}_i^{(in)} = \sum_j \mathbf{p}_j^{(out)} \tag{14}$$

and the circle in the middle represents some potentially complicated bunch of local interactions.

Interactions describe *bound states* (like a hydrogen atom) or *scattering (unbound) states.* To probe a bound state, we can study its absorption of a photon: $\gamma A \to A^*$, where A^* is an excited state; this is a $2 \to 1$ process. Or, emission: $A^* \to \gamma A$, a $1 \to 2$ process.

Interactions can be classified as:

- **Elastic collisions:** The masses of the particles are unchanged; eg, $A + B \to A + B$, a $2 \to 2$ process
- **Inelastic collisions:** The masses change — kinetic energy and mass are interchanged — and new particles may even be formed; eg, $A + B \to C + D + \cdots$, a $2 \to N$ process.
- **Absorption:** Eg, $A + B \to C$, a $2 \to 1$ process; the masses must change.
- **Emission:** Eg, $A \to B + C$, a $1 \to 2$ process; the masses must change, but usually, A and B are different states of the same atom.
- **Decay:** Eg, $A \to B + C + \cdots$, a $1 \to N$ process; the masses must change.

Digression on the Standard Model of Particle Physics

At high energies, these local interactions are well understood, and fully described by the *Standard Model of Particle Physics.* This theory is built in the language of Quantum Field Theory (QFT), which brings together quantum mechanics, relativity, and field theory (generalization of Maxwell field theory). We also need Lie Group theory and Renormalization Group theory to fully specify and understand it. Space-time symmetries are built in from the start: homogeneity, isotropy, Lorentz invariance, and three discrete symmetries (C,P,T). All interactions occur at space-time events with no extent (ie, local). At some level, the math is elegant and straightforward; but in any real calculation, gets very complicated, grungy, and inelegant, very fast. The underlying concepts are very deep and profound!

The Standard Model does not (yet) include gravity; we do not yet have a complete understanding of quantum gravity. It does include the three other known fundamental interactions of nature:

- Electromagnetism and the interaction of light with charged matter: Quantum Electrodynamics, QED; the interaction is mediated by the massless photon.
- The strong nuclear interaction between quarks: Quantum Chromodynamics (QCD); the interaction is mediated by eight massless gluons.
- The weak nuclear interaction: Quantum Flavor Dynamics (QFD): the interaction is mediated by three massive particles: W^+, Z^0, W^-. This interaction maximally violates the discrete symmetries P (parity inversion) and T (time reversal), but conserves the product PT as well as CPT.

The description of these interactions has been extremely well-tested over the last 50 years and is now very well-established. Other theories at the frontiers of our understanding (and just beyond experiment) include:

- Grand Unification brings all three theories together, but experiment has yet to find any direct evidence of it (like proton decay).

- One more space-time symmetry remains: Supersymmetry (SUSY). It is much beloved by theorists but experiment has yet to find any evidence for it (but stay tuned!). One manifestation would be the Lightest Supersymmetric Particle (LSP), which is a perfect candidate for dark matter.
- String theory gives up locality; strings are fundamental objects that are extended in space-time. It would be a consistent quantum theory of gravity. If string theory is relevant in the real world, it requires SUSY.

Example of elastic collisions – Compton scattering: (This is Hand and Finch problem 12-17a.) A gamma ray photon scatters off a stationary electron. How does the outcoming energy/frequency of the photon depend on the incoming energy/frequency and the scattering angle θ'? See Hand and Finch for a sketch.

One scheme for solving the problem is the following. Four momentum conservation gives

$$\mathbf{p}_\gamma^{(in)} + \mathbf{p}_e^{(in)} = \mathbf{p}_\gamma^{(out)} + \mathbf{p}_e^{(out)}. \tag{15}$$

We are not asked for the details of $\mathbf{p}_e^{(out)}$ and so we rewrite this equation as

$$\mathbf{p}_\gamma^{(in)} - \mathbf{p}_\gamma^{(out)} + \mathbf{p}_e^{(in)} = \mathbf{p}_e^{(out)} \tag{16}$$

so that on taking the magnitude squared of both sides (in the 4-vector sense)

$$(\mathbf{p}_\gamma^{(in)} - \mathbf{p}_\gamma^{(out)})^2 + 2\mathbf{p}_e^{(in)} \cdot (\mathbf{p}_\gamma^{(in)} - \mathbf{p}_\gamma^{(out)}) + (\mathbf{p}_e^{(in)})^2 = (\mathbf{p}_e^{(out)})^2. \tag{17}$$

Now using $\mathbf{p}_e^2 = m_e^2, \mathbf{p}_\gamma^2 = 0$ this simplifies to

$$\mathbf{p}_e^{(in)} \cdot (\mathbf{p}_\gamma^{(in)} - \mathbf{p}_\gamma^{(out)}) = \mathbf{p}_\gamma^{(in)} \cdot \mathbf{p}_\gamma^{(out)} \tag{18}$$

and $\mathbf{p}_e$ drops out of the problem. Using $\mathbf{p}_e^{(in)} = (m_e, 0), \mathbf{p}_\gamma^{(in)} = E_\gamma^{(in)}(1, \hat{x}), \mathbf{p}_\gamma^{(out)} = E_\gamma^{(in)}(1, \hat{n})$ with $\hat{n}$ the direction of propagation of the outgoing photon and $\cos\theta' = \hat{x} \cdot \hat{n}$ gives

$$m(E_\gamma^{(in)} - E_\gamma^{(out)}) = \mathbf{p}_\gamma^{(in)} \cdot \mathbf{p}_\gamma^{(out)} = E_\gamma^{(in)} E_\gamma^{(out)} (1 - \cos\theta'). \tag{19}$$

This gives the expression in Hand and Finch, but a more useful form is

$$\frac{1}{E_\gamma^{(out)}} - \frac{1}{E_\gamma^{(in)}} = \frac{1}{m_e}(1 - \cos\theta'), \tag{20}$$

or using $E_\gamma = h\nu = h/\lambda$ (remember $c = 1$)

$$\lambda^{(out)} - \lambda^{(in)} = \frac{h}{m_e}(1 - \cos\theta'). \tag{21}$$

Example on inelastic collisions – Colliding putty balls: Consider a ball of mass m_a moving with velocity $\vec{u}_a$ colliding with a stationary ball of mass m_b. After the collision the balls stick together and move with velocity $\vec{u}$. What is the mass m of the new ball, and what is $\vec{u}$?

It is easy enough to write down the total relativistic 3-momentum and energy, and equate the before and after values. However it is often simpler to manipulate the 4-momenta directly. With obvious notation, using the conservation law $\mathbf{P}_{\text{out}} = \mathbf{P}_{\text{in}}$ with $\mathbf{P}$ the total 4-momentum (sum of the individual particle 4-momenta), we have

$$\mathbf{P}_{\text{out}}^2 = m^2 = \mathbf{P}_{\text{in}}^2 = (\mathbf{p}_a + \mathbf{p}_b)^2 = \mathbf{p}_a^2 + \mathbf{p}_b^2 + 2\mathbf{p}_a \cdot \mathbf{p}_b = m_a^2 + m_b^2 + 2E_a m_b, \tag{22}$$

where in the last step we have used $\mathbf{p}_b = (m_b, 0)$ so that only the time-like component of $\mathbf{p}_a$, the energy E_a of particle a, appears in the scalar product. This gives

$$m = \sqrt{m_a^2 + m_b^2 + 2E_a m_b} \geqslant m_a + m_b. \tag{23}$$

The velocity is given by

$$\vec{u} = \frac{\vec{P}_{\text{out}}}{E_{\text{out}}} = \frac{\vec{P}_{\text{in}}}{E_{\text{in}}} = \frac{\gamma_a m_a \vec{u}_a}{\gamma_a m_a + m_b}, \tag{24}$$

here using the conservation of the total 3-momentum and of the total energy in the second step.

Center of momentum frame

An important general approach (not always necessary in simple cases) is to transform to the frame in which the total 3-momentum $\vec{P}$ is zero. This is analogous to the center of mass frame in Newtonian physics. The collision is particularly simple in this frame: in a binary collision, the incoming particles must have equal and opposite 3-momenta, and the same applies for the outgoing particles if there remain only two.

The general scheme is

- Choose the x-direction along the total 3-momentum so that $P_y = P_z = 0$, and we can use our standard configuration for Lorentz transforming from the lab frame S to the center of momentum frame S';
- Since $P'_x = 0$, Eq. (4) shows that the speed of S' relative to S is $v = P_x/E$;
- Transform the energies and 3-momenta of all the particles to S' using the Lorentz transformation with the speed v;
- Solve the collision in S' (outgoing energies and momenta etc.);
- Transform back to S.

This method is direct and always works. The results can also be found using the invariance of scalar products such as $\mathbf{p}_a \cdot \mathbf{p}_b = \mathbf{p}'_a \cdot \mathbf{p}'_b$ with $\mathbf{p}_a, \mathbf{p}_b$ and $\mathbf{p}'_a, \mathbf{p}'_b$ any two of the particle 4-momenta in the lab frame and in the CM frame, respectively. This approach leads to less algebra but requires more ingenuity.

Example - Threshold energy to produce new particles: This is a very important problem in particle physics. Consider the production of proton-antiproton pairs by colliding a beam of protons with a target of stationary protons: $p + p \to p + p + \bar{p} + p$. What is the minimum kinetic energy T of the beam particles for the production to occur? You might guess $T = 2m_p$ with just enough kinetic energy to convert to two proton masses. In fact the answer is significantly larger.

Call the incoming beam proton a and the target proton b, so that $\mathbf{p}_a = (E_a, \vec{p}_a)$ and $\mathbf{p}_b = (m_p, 0)$. The magnitude of $\vec{p}_a$ is given from $mathbfp_a \cdot \mathbf{p}_a = m_p^2 = E_a^2 - ||\vec{p}_a|^2|$. In the center of momentum frame the threshold is when all the outgoing particles are at rest (of course this is consistent with zero total 3-momentum). This means that the outgoing 4-momentum is $\mathbf{p'}_{\text{out}} = (4m_p, 0)$. In the COM frame the incoming 4-momenta are $\mathbf{p}'_a = (2m_p, \vec{p}^*)$ and $\mathbf{p}'_b = (2m_p, -\vec{p}^*)$. Since $\mathbf{p}' \cdot \mathbf{p}' = m_p^2 = (2m_p)^2 - |\vec{p}^*|^2$, we know that $|\vec{p}^*|^2 = 3m_p^2$. Now use the Lorentz invariance of the scalar product, evaluated in the lab (unprimed) and COM (primed) frames: $\mathbf{p}_a \cdot \mathbf{p}_b = \mathbf{p}'_a \cdot \mathbf{p}'_b$. We get:

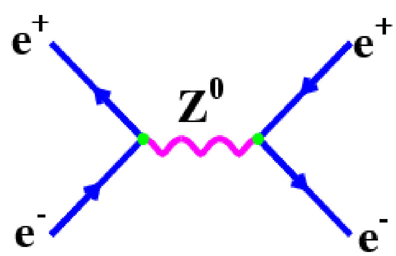
$$E_a m_p = 4m_p^2 + (\vec{p}^*)^2 = 4m_p^2 + (4m_p^2 - m_p^2) = 7m_p^2 \tag{25}$$

using Eq. (1) in the second step. Thus $E_a = 7m_p$ and the threshold kinetic energy is $6m_p$, three times larger than the naive guess.

The general expression for the threshold energy for the binary collision between a beam of particles of mass m_a colliding with a target of stationary particles of mass m_b to produce outgoing particles with total mass $\sum m_{\text{out}}$ is

$$E_a|_{\text{threshold}} = \frac{(\sum m_{\text{out}})^2 - m_a^2 - m_b^2}{2m_b}. \tag{26}$$

The result is even more dramatic for the production of the Z^0 boson (mass 91 GeV) particle by colliding positrons (anti-electrons) with electrons (both have mass $m_e = 0.5$ MeV):



$$e^+ + e^- \to Z^0 \to e^+ + e^-$$

We want the *invariant mass* of the e^+e^- system, which is the energy in the COM frame E_{cm}, to be the mass of the Z^0. More generally, in the reaction $a + b \to c$ where a is the beam particle, b is the target,

$$E_a = \frac{m_c^2 - m_a^2 - m_b^2}{2m_b}. \tag{27}$$

Here, the invariant mass squared is $m_c^2 = E_{cm}^2 = m_Z^2$. In particle physics jargon, $E_{cm}^2 = s$, where s is a *Mandelstam variable.* We can try using a beam positrons and a stationary electron target; we'll need $E_{e^+} = 10^{10}$ MeV, which is way beyond the reach of accelerator technology. The solution is to use *two colliding beams*, with $E_{e^+} = E_{e^-} = m_Z/2 = 45.5$ GeV - which puts us in the center of momentum frame for symmetric collisions. The LEP collider at CERN ran with these energies in the 1990's.

Relativistic Force

The 4-force or Minkowski force takes the form

$$\mathbf{f} = \frac{d\mathbf{p}}{d\tau} \tag{28}$$

(4-vector $d\mathbf{p}$ times scalar $d\tau^{-1}$). In some inertial frame where the particle velocity is $\vec{u}$ this gives

$$\mathbf{f} \to (\gamma_u \frac{dE}{dt}, \gamma_u \frac{d\vec{p}}{dt}) \ . \tag{29}$$

There are some nice formal expressions involving the 4-force, such as the expression for the electromagnetic 4-force

$$\mathbf{f} = q\mathbf{F} \cdot \mathbf{u} \tag{30}$$

with $\mathbf{F}$ the electromagnetic field tensor and $\mathbf{u}$ the particle 4-velocity, but more often we focus on the relativistic 3-force.

It turns out that the most physical definition of the relativistic 3-force is

$$\vec{f} = \frac{d\vec{p}}{dt}. \tag{31}$$

Using $\mathbf{p}^2 = m^2$ gives

$$\frac{d\mathbf{p}^2}{d\tau} = 0 = 2\mathbf{p} \cdot \frac{d\mathbf{p}}{d\tau} = 2m\mathbf{u} \cdot \mathbf{f} \tag{32}$$

so that $\mathbf{u} \cdot \mathbf{f} = 0$ (the 4-force is always "perpendicular" to the 4-velocity). Evaluating this in an inertial frame where $\mathbf{u} = \gamma_u(1, \vec{u})$ and $\mathbf{f} = \gamma_u(dE/dt, \vec{f})$ gives

$$\frac{dE}{dt} = \vec{f} \cdot \vec{u} \tag{33}$$

so that the 3-force defined by Eq. (31) has the desired connection with the energy. We will see that for a charged particle in electric and magnetic fields, the expression for $\vec{f}$ is the usual Lorentz force. Thus Eq. (31) is the most useful definition of the 3-force. Note that $\gamma_u\vec{f}$ is the spacelike component of the force 4-vector: the transformation behavior of $\vec{f}$ between inertial frames can be obtained from this.

Although Eq. (31) is an expression analogous to Newton's law, force=mass $\times$ acceleration is no longer true. In general the acceleration $\vec{a} = d\vec{u}/dt$ is not even parallel to the force:

$$\vec{f} = \frac{d}{dt}(m\gamma_u\vec{u}) \tag{34}$$

$$= m\gamma_u^3(\vec{u} \cdot \vec{a})\vec{u} + \gamma_u m\vec{a} \tag{35}$$

For forces parallel and perpendicular to $\vec{u}$ the result is simpler

$$f_{\|} = \gamma^3 m a_{\|} \tag{36}$$

$$\vec{f}_{\perp} = \gamma m\vec{a}_{\perp} \tag{37}$$

but with different proportionality constants (sometimes called longitudinal and transverse masses). For other directions $\vec{f}$ and $\vec{a}$ are not parallel.

Example – motion under constant force: Consider a constant force $\vec{f} = f\hat{x}$. The equation of motion is

$$\frac{dp_x}{dt} = f \qquad \text{so that } p_x = ft\ . \tag{38}$$

The velocity in the x-direction is most easily calculated from

$$u_x = \frac{p_x}{E} = \frac{p_x}{\sqrt{p_x^2 + m^2}} \tag{39}$$

to give

$$u_x = \frac{ft}{\sqrt{m^2 + (ft)^2}}. \tag{40}$$

The velocity approaches 1 (the speed of light) in the long time limit.

Lagrangian Formalism

I discussed the relativistic Lagrangian of a free particle and a particle in an electromagnetic field in Lecture 2 of Ph106a. The argument might make more sense now, so I will outline it again here.

The Lagrangian approach is given by minimizing the action. We make the bold guess that the action is a Lorentz invariant (same in all inertial frames). For a single particle of mass m we can write it as the integral along the worldline parameterized by the proper time between fixed beginning and ending events $\mathcal{P}_1$ and $\mathcal{P}_2$

$$S = \int_{\mathcal{P}_1}^{\mathcal{P}_2} \mathcal{L}\, d\tau \; . \tag{41}$$

For different physical situations we must choose a physically sensible Lorentz invariant expression for $\mathcal{L}$: this is a strong constraint and the appropriate expression (up to unimportant multiplicative and additive constants) is often unique.

Evaluating in our inertial frame gives

$$S = \int_{t_1}^{t_2} L\, dt = \int_{t_1}^{t_2} (\gamma L)\, d\tau \tag{42}$$

where the factor $\gamma \equiv \gamma_u$ with $\vec{u}$ the instantaneous speed of the particle comes from the $dt \to d\tau$ transformation for our time to proper time. Thus γL is Lorentz invariant.

Free particle

The only invariant that we can use is a constant for the particle. (Check that others don't make sense, e.g. $\mathbf{x}^2$ would give an origin dependent result, $\mathbf{u}^2 = 1$ and so does not give anything nontrivial, ...). Matching to the Newtonian expression for small speeds gives $\gamma L = -m$ with m the mass so that

$$L = -m\sqrt{1-u^2} \; . \tag{43}$$

This gives for small u

$$L \to -m + \frac{1}{2} m u^2 \tag{44}$$

the correct Newtonian result (additive constants do not affect the results derived from the Lagrangian). Also, we can compute the conjugate momentum (from Hamiltonian dynamics):

$$p_x = \frac{\partial L}{\partial u_x} = \frac{m u_x}{\sqrt{1-u^2}} = \gamma_u m u_x \tag{45}$$

or doing all three components together:

$$\vec{p} = \frac{\partial L}{\partial \vec{u}} = \frac{m\vec{u}}{\sqrt{1-u^2}} = \gamma_u m \vec{u}, \tag{46}$$

and we recover the 3-momentum expression.

The Hamiltonian is

$$H = \vec{p}\cdot\vec{u} - L = \gamma m = E, \tag{47}$$

and is equal to the energy even though the kinetic energy is not a quadratic form in $\vec{u}$. Note that the Hamiltonian is *not* a Lorentz invariant quantity; like L, it is evaluated in a specific frame.

Charge in electromagnetic field

We look for a Lagrangian linear in the fields and depending on the particle velocity. It turns out to be convenient to use the scalar and vector potentials $\Phi, \vec{A}$, since $(\Phi, \vec{A})$ form a 4-vector, rather than $\vec{E}, \vec{B}$. The only new Lorentz invariant (with the right symmetry properties, linear in the field strength etc.) is a constant times the scalar product $\mathbf{u} \cdot \mathbf{A}$ where $\mathbf{u}$ is the *velocity 4-vector* and $\mathbf{A}$ is the *electromagnetic potential 4-vector*. Thus for a particle in an electromagnetic field

$$\mathcal{L} = -m - q\mathbf{u} \cdot \mathbf{A} = \gamma L$$

where again the constant is chosen to match onto Newtonian physics at small velocities: q is then the charge of the particle. In our inertial frame, the components of the 4-vectors are $\mathbf{u} = \gamma(1, \vec{u}), \mathbf{A} = (\Phi, \vec{A})$ with Φ the electric scalar potential and $\vec{A}$ the magnetic vector potential. Thus the Lagrangian in our frame of reference is

$$L = -m\sqrt{1-u^2} - q\Phi(\vec{x}, t) + q\vec{u} \cdot \vec{A}(\vec{x}, t).$$

Note how the combination $\Phi - \vec{u}.\vec{A}$ is forced on us by the requirement of Lorentz invariance.

From this Lagrangian we can derive the Lorentz force. The canonical 3-momentum is

$$\vec{p} = \frac{\partial L}{\partial \vec{u}} = m\gamma\vec{u} + q\vec{A}. \tag{48}$$

The first term on the right hand side is the same as for the free particle; it is the *kinetic momentum* associated with the particle motion. The second term, from the presence of an electromagnetic vector potential, demonstrates that the canonical 3-momentum may not be what we usually think of as momentum. The kinetic momentum is $\vec{\pi} = m\gamma\vec{u} = (\vec{p} - q\vec{A})$. The Euler-Lagrange equation is

$$\frac{d}{dt}\frac{\partial L}{\partial \vec{u}} = \frac{d\vec{p}}{dt} = \frac{\partial L}{\partial \vec{x}}. \tag{49}$$

This gives

$$\vec{f} = \frac{d\vec{\pi}}{dt} = q\left[-\vec{\nabla}\Phi + \vec{\nabla}(\vec{u} \cdot \vec{A}) - \frac{d\vec{A}}{dt}\right], \tag{50}$$

$$= q\left[-\vec{\nabla}\Phi + \vec{u} \times (\vec{\nabla} \times \vec{A}) + (\vec{u} \cdot \vec{\nabla})\vec{A} - \frac{d\vec{A}}{dt}\right], \tag{51}$$

We used a standard vector equality in the last identity in order to pull $\vec{u}$ out of the gradient. Now $d\vec{A}/dt$ is the *total* time derivative of the vector potential moving with the particle

$$\frac{d\vec{A}}{dt} = \frac{\partial \vec{A}}{\partial t} + (\vec{u} \cdot \vec{\nabla})\vec{A}, \tag{52}$$

and from the definitions of the scalar and vector potential

$$\vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}, \qquad \vec{B} = \vec{\nabla} \times \vec{A}, \tag{53}$$

so that the Lorenz force $\vec{f} = d\vec{\pi}/dt = d(m\gamma\vec{u})/dt$ is

$$\vec{f} = q(\vec{E} + \vec{u} \times \vec{B}) \ . \tag{54}$$

This is the familiar Lorentz force law.

The Hamiltonian is:

$$H = \vec{p} \cdot \vec{u} - L = \gamma m + q\Phi = W, \tag{55}$$

where we are now using W for the energy to remove confusion with the electric field magnitude E.

Example – Charge in uniform $\vec{B}$: The equations of motion are

$$\frac{d\vec{\pi}}{dt} = \frac{d(\gamma m\vec{u})}{dt} = q\vec{u} \times \vec{B}\,, \qquad \frac{dW}{dt} = 0 \tag{56}$$

The second equation gives γ constant, so that $|\vec{u}|$ is constant, and then the first equation is

$$\frac{d\vec{u}}{dt} = \vec{u} \times \vec{\omega}_B \qquad \text{with} \qquad \vec{\omega}_B = \frac{q\vec{B}}{\gamma m}. \tag{57}$$

This is circular motion in the plane perpendicular to $\vec{B}$ at the cyclotron frequency ω_B, together with constant velocity along $\vec{B}$.