Physics 106a, Caltech — 21 November, 2019

Lecture 16 – Principle of relativity, space-time diagrams, Lorentz transformation

Outline of Special Relativity — (4 lectures, Hand and Finch Chapter 12):

- Principle of relativity, space-time diagrams, Lorentz transformation
- 4-vectors, space-time invariants, geometrical picture
- Energy-momentum 4-vector, relativistic Lagrangian for particles and for EM fields
- Lorentz boosts, Lorentz forces

Principle of Relativity

The Principle of Relativity states:

> The laws of physics are the same in all inertial frames.

In Special Relativity *inertial frames* are a special set of frames in which the laws of physics are particularly simple (rotating frames are *not* simple, since "fictitious" apparent forces like centrifugal and Coriolis, arise), with any two frames in the set moving at a constant relative velocity. In General Relativity, an inertial frame is any *free floating* frame, and includes frames freely falling under gravity.

We will consider two frames, S and S', wherein observers at rest in their respective frames are moving with respect to one another by a constant velocity $\vec{v} = v\hat{x}$. Their spatial coordinate systems look like Fig. 1:

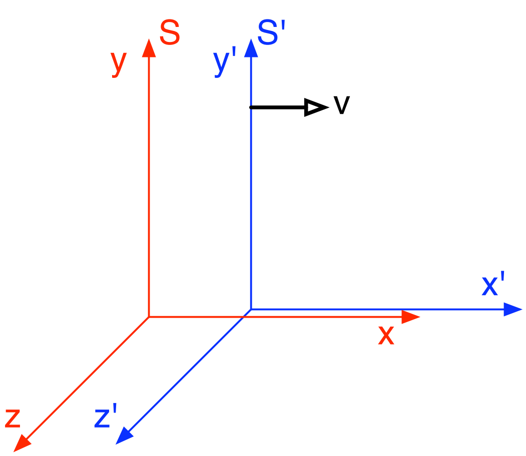

Figure 1: 3D Spatial axes in frame S and frame S' moving with respect to S with constant velocity $v\hat{x}$.

Galilean relativity

The principle is the same as the Galileo-Newton Principle of Relativity. Non-relativistic Newtonian physics is invariant (laws of physics are the same in different inertial frames) under Galilean transforms from $S \to S'$:

$$
\begin{aligned}
x' &= x - vt \\
y' &= y \\
z' &= z \\
t' &= t
\end{aligned}
$$

From this we can compute the velocity of a particle in frame S: $u = dx/dt$, and in frame S': $u' = dx'/dt' = u - v$.

Enter Maxwell

As the laws of electromagnetism became understood, it appeared that these were not consistent with the principle, since Maxwell's equations predict a wave propagating with a speed c that can be determined by static measurements: $c = 1/\sqrt{\varepsilon_0 \mu_0}$, where ε_0 is the *vacuum electrostatic permeability* that appears in Coulomb's Law, and μ_0 is the *vacuum magnetostatic permeability* that appears in Ampere's Law.

With the Newtonian understanding of space and time, if a speed is c in one inertial frame S, then it would be a different speed in another inertial frame S' (see Galilean relativity, above).

Thus either there is a *special* frame (the "ether") in which Maxwell's equations are true, and the Principle of Relativity fails for electromagnetism, or the Principle of Relativity holds for electromagnetism, in which case the Newtonian notion of space-time must fail. In response to the failure of the Michaelson-Morley experiment to detect the ether, Einstein proposed the latter. To emphasize this a second postulate of Special Relativity is often added:

> All laws of physics, including E&M, are the same in inertial frames, and in particular, the speed of light is the same in all inertial frames.

A consequence, as we will see, is that time is no longer absolute. For example, spatially separated events that are simultaneous in one inertial frame need not be simultaneous in a different inertial frame. One way to proceed is to see how the description of space and time transform between different inertial frames. This introduces the *Lorentz transformation.* In the next lecture we will proceed more geometrically.

Lorentz Transformation

How are space coordinates (x, y, z) and time t in one inertial frame S of reference related to space coordinates (x', y', z') and time t' in a second inertial frame S'? In other words, how can we modify the Galilean transform to ensure that the speed of light is the same in all inertial frames? First we establish what we mean by coordinates.

Coordinates

In our inertial frame S we imagine setting up space-time coordinates from a lattice of rulers and synchronized clocks. I show this in Fig. 2 for just one space dimension. The clocks can be synchronized in a number of ways, just as we usually do. I could purchase a large number of identical

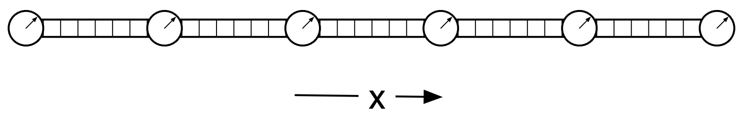

Figure 2: Lattice of rulers and clocks for one dimension

synchronized clocks from China, and transport them to the various sites. I might worry about time dilation effects if I move them too rapidly, but these can always be made negligible by transporting the clocks slowly enough. Or I could send a radio signal from a central site, and include the correction for the travel time at the speed of light. Or for a pair of clocks I could send a flash of light from a point midway between, and set the clocks to some agreed time when the flashes arrive.

We use this lattice of rulers and clocks to measure the coordinates (x, y, z) and time t in the frame S of a definite point in space-time. We call a precise space-time point an *event*. As the name implies, it is conceptually useful to think of a definite physical event as defining a space-time point, e.g. an atom emits a flash of light, or I clap my hand[1]. It important to think about physical processes involving things happening at different space-time points in terms of events making up the process.

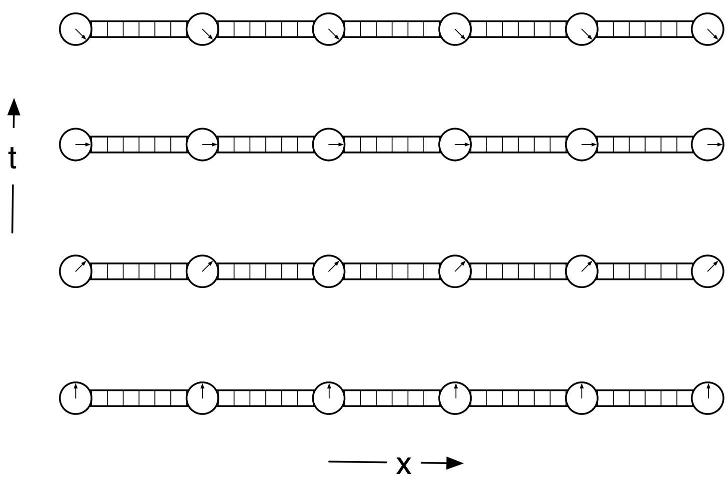

Figure 3: The same lattice of rulers and clocks for 4 successive times

Figure 3 shows the rulers and clocks for four successive times. The coordinate of an event is given by the ruler division and the clock reading at the event. Plotting the coordinates of a sequence of events gives a space-time diagram. Usually, of course, we will just use x, t axes, and not draw clocks and rulers!

[1]Obviously the "spread" of the physical event must be small compared to the space-time resolution desired.

Units

Our conventional unit of time (second) is defined as the time for 9192631770 oscillations of radiation corresponding to the transition between the two hyperfine levels of the ground state of the Cs^{133} (caesium 133) atom, since these are amongst the most stable *clocks* that can be made in the laboratory. Our conventional unit of length (meter) is defined as $1/(2.99792458\times10^8)$ of the distance traveled by electromagnetic radiation in one of these seconds. The speed of light is therefore, by definition of our units, $c = 2.99792458 \times 10^8$ meters/second. In setting up the mathematical description of the physics it makes little sense to include these numbers arrived at by historical accident. We will instead, from now on, pick some convenient unit for time, e.g. the time for 1 oscillation of the Cs radiation, and then choose as the units for distances (x, y, x) the distance traveled by the electromagnetic radiation (in vacuum) in this time. With these choices the speed of light $c = 1$, and the symbol c will not appear in any of the expressions. To return to conventional units, factors of c are inserted to give correct dimensions.

Space-time diagrams

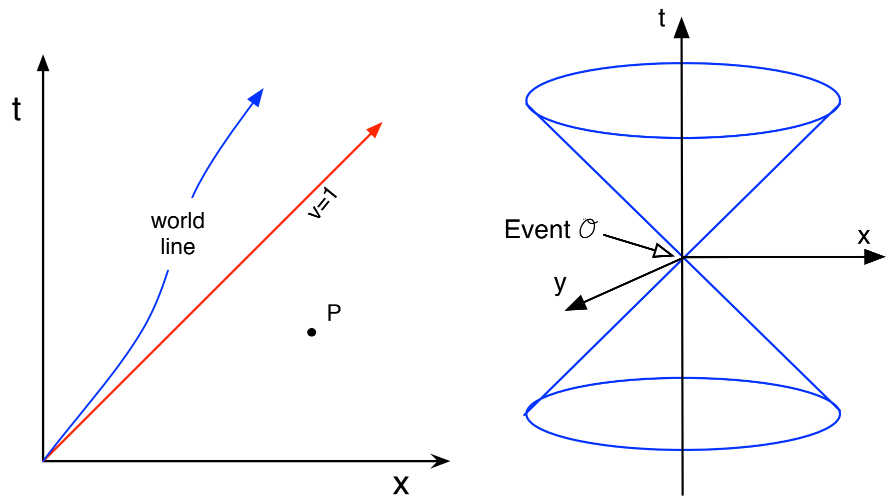

Figure 4: Left: Space-time diagram for one space dimension (x) and one time dimension (t), showing an event $\mathcal{P}$ at spacetime coordinates $(x_{\mathcal{P}}, t_{\mathcal{P}})$, the worldline of a particle (blue), and the worldline of a photon (red). Right: Space-time diagram for two space dimensions (x, y) and one time dimension (t), showing an event's forward and backward light cones.

An event is represented by a point in a four dimensional space, three space dimensions and time. To represent this pictorially we often use a graph with time as the ordinate, and one (or sometimes two) space coordinates as abscissa(s). An example is shown in Fig. 4 left, for x and t. The plot shows an event $\mathcal{P}$, and the worldline of a particle moving in the x-direction — the line joining all the events at which the particle is present. The worldline of an observer at rest in that frame, at $x = 0$, coincides with the t axis. Also shown is the worldline of a pulse of light or a photon which moves at the speed of light. For the relativistic choice of units, the slope of the photon worldline is unity; a photon worldline appears as a 45° line in this diagram.

The worldline of a particle moving in the x-direction with velocity $u < c$ has an instantaneous slope of $1/u > 1$ in relativistic units. At every point along the particle's worldline, the velocity and thus the slope may change. At every point, one can draw a *forward light cone* at $\pm 45°$; the worldline must stay within that cone in order for its speed to remain $u < 1$. We refer to it as a *cone* to remind us that there are two other spatial directions, not just $\hat{x}$. If we also include $\hat{y}$ (and still suppress $\hat{z}$), you can see in Fig. 4 right, that it is shaped like a cone.

Standard configuration for frames, and three events

We consider a "standard configuration" for our two frames of reference S with coordinates t, x, y, z and S' with coordinates (t', x', y', z'): the coordinate axes are aligned, and the origins of the axes coincide at $t = t' = 0$. The frame S' moves at speed v along the x-axis relative to S (and so the frame S moves at the speed v along the negative x' axis relative to S'); see Fig. 1. Thus the event "the origins coincide" has the coordinates $x = y = z = t = 0$ in S and $x' = y' = z' = t' = 0$ in S'. If an event $\mathcal{P}$ has coordinates t, x, y, z in S, what are the coordinates in S'? You have done this in Ph 1, and one derivation is worked out in Hand and Finch §12.1-4. Here is a brief review of the argument.

To relate the two frames, consider a series of three events: a pulse of light is (1) emitted from a source with an observer at rest in the S' frame, towards a mirror at rest at $x' = L$, is (2) reflected, and (3) returns to the observer at rest in S'. This is shown in the S' frame on the left side of Fig. 5.

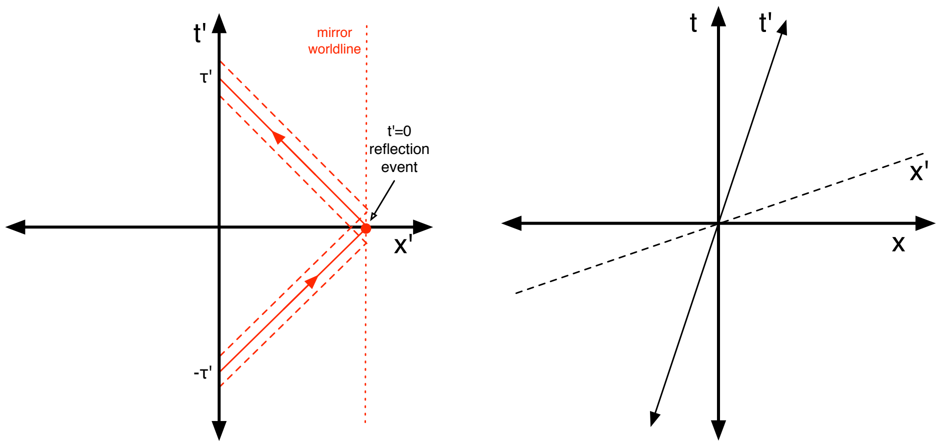

Figure 5: Left: In the S' frame, a photon is emitted at $(x', t') = (0, -\tau')$, is reflected at $(L, 0)$; and is received at $(0, \tau')$. The photon path is at $\pm 45°$. Right: As a first step to understanding these three events in the S frame, we draw both the S and S' (x, t) axes on the same plot.

Analyzing the three events in the S frame

We then construct the following argument:

- The coordinates transverse to the direction of relative motion are unchanged $y = y', z = z'$ (see Fig. 1). We now just look at x and t.
- Throughout, keep in mind that in these spacetime diagrams, you cannot naively apply trigonometry, or the Pythagorean theorem, as you might for spatial coordinates.

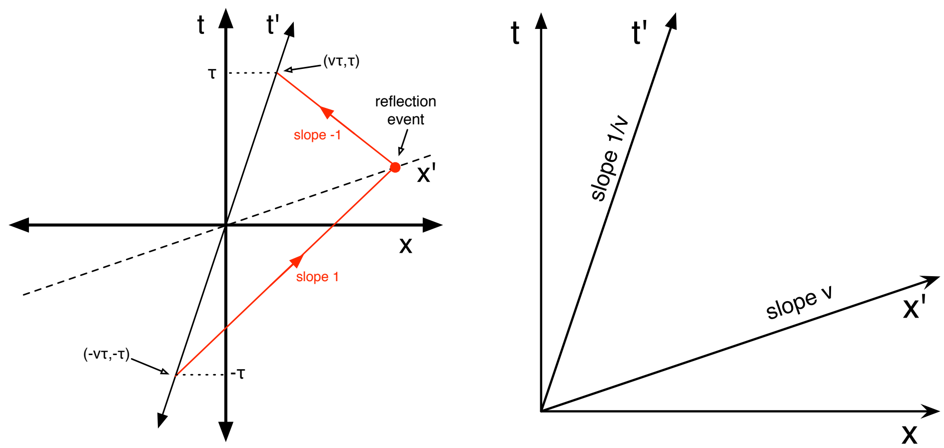

Figure 6: Left: In the S frame (orthogonal (x, t) axes), we draw the S' frame (x', t') axes; the photon emission, reflection and reception events, and the photon worldlines (with slopes ± 1) that connect them. Right: x' axis ($t' = 0$) and t' axis ($x' = 0$) in S frame space-time diagram.

- We can draw our three events in the S' frame, as shown in Fig. 5, Left.
- The coordinate origin $x' = 0$ in S' (i.e. the t' axis) is described by the coordinates $x = vt$ in S by definition of the two frames. Thus, we can draw the t' axis on the S frame spacetime diagram, with a slope of $1/v > 1$, as shown in Fig. 5, Right.
- At this point, we don't know where to draw the x' axis. But we know that the two frames are related by a constant velocity, so everything should be linear: the x' axis should be a straight line (going through $(x, t) = (0, 0)$, where the two frames coincide), with some yet-to-be-determined slope. Let's draw it somewhere, as in Fig. 5, Right.
- Next, we can add the photon emission and reception events, which lie along the t' axis, at $\pm\tau'$ (see Fig. 5 Left).
- Next, note that since the t' axis has a slope of v in the S frame, the coordinates of the photon emission and reception events in the S frame are $(x, t) = (\mp v\tau, \tau)$, where τ is some constant that can be related to τ' (but we won't need to).
- Next, we know that the reflection event lies somewhere on the x' axis. That event is connected to the emission event by a photon worldline with slope 1 (ie, at $+45°$) *in both the S and S' frames, according to the Principle of Relativity.* So, draw that photon worldline from the emission event until it hits the x' axis (the slope of which we still do not know); this must be the location of the reflection event.
- Draw the photon worldline (with slope of -1, ie, at $-45°$) from the reflection event to the reception event on the t' axis. The diagram should now look like Fig. 6 Left.
- Now we can relate the (x, t) coordinates (in the S frame) for the three events as follows:

  $$(-v\tau, -\tau) + r(1, 1) + s(-1, 1) = (v\tau, \tau) \quad \Rightarrow r = (1 + v)\tau, s = (1 - v)\tau$$

  where r and s are two parameters that we have now determined, in terms of τ.

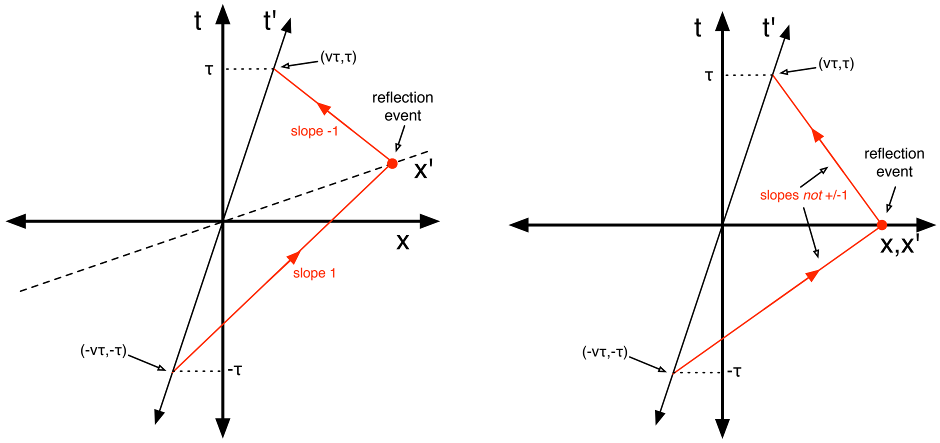

Figure 7: Left: Same as Fig. 6 Left. Right: This is how it would look for a Galilean transform; there would be no way to get the photon worldlines connecting the three events to have slopes of ± 1 in the S frame.

- This means that the reflection event is at

$$(-v\tau, -\tau) + r(1,1) = (-v\tau, -\tau) + (1+v)\tau(1,1) = (\tau, v\tau)$$

  which means that the x' axis has a slope of v. We have now completely determined the x' axis in the S frame spacetime diagram. The S' axes are shown in the S space-time diagram in Fig. 6 Right. Note that we don't need τ, τ', r or s anymore. They all scale with L, which we can choose to have any value.

- Note that if we tried to do this with a Galilean transform, we would not be able to force the two photon worldlines to have slopes of ± 1; see Fig. 7 Right.

Constructing the Lorentz transforms from S to S' and back

- We have established that in the S frame, the t' axis has a slope of $1/v$, and the x' axis has a slope of v. They are both straight lines, so the relationship between (x, t) and (x', t') is linear.

- Thus, a point on the t' axis ($x' = 0$), in the S frame, is proportional to the vector $(x, t) \sim (v, 1)$; and a point on the x' axis ($t' = 0$), in the S frame, is proportional to the vector $(x, t) \sim (1, v)$.

- An arbitrary point (x', t') must therefore have coordinates in the S frame of

$$(x, t) = \gamma(v)(1, v)x' + \tilde{\gamma}(v)(v, 1)t' = (\gamma(v)x' + \tilde{\gamma}(v)vt', \gamma(v)vx' + \tilde{\gamma}(v)t'),$$

  where $\gamma(v)$ and $\tilde{\gamma}(v)$ are two constants of proportionality (which depend on the constant velocity v), to be determined.

- The transformation $S \to S'$ is the same as the transformation $S' \to S$ with $v \to -v$. Thus,

$$(x', t') = (\gamma(-v)x - \tilde{\gamma}(-v)vt, -\gamma(-v)vx + \tilde{\gamma}(-v)t).$$

- Now we will argue that $\gamma(-v) = \gamma(v)$ and $\tilde{\gamma}(-v) = \tilde{\gamma}(v)$, ie, these so-far unknown quantities are *even* functions of v. Consider our photon emission and reception events, at $t' = \mp\tau'$, lying on the t' axis at $x' = 0$, at $(x, t) = (\tilde{\gamma}(v)vt', \tilde{\gamma}(v)t')$.

- Now reverse the sign of v. Our photon emission and reception events in that case are at $(x, t) = (-\tilde{\gamma}(-v)vt', \tilde{\gamma}(-v)t')$. From Fig. 6 Left, we see that this just flips the diagram about the t (vertical) axis, and the slope of the new t' axis simply flips sign. We can reconcile these two cases only if $\gamma(-v) = \gamma(v) = \gamma(|v|)$ and $\tilde{\gamma}(-v) = \tilde{\gamma}(v) = \tilde{\gamma}(|v|)$.

- We see that the transformation $S \to S'$ must be linear in x, t

$$x' = \gamma x - \tilde{\gamma} vt, \qquad t' = \tilde{\gamma} t - \gamma vx$$

  with $\gamma = \gamma(|v|), \tilde{\gamma} = \tilde{\gamma}(|v|)$.

- The inverse transformation $S' \to S$ is given by $v \to -v$:

$$x = \gamma x' + \tilde{\gamma} vt', \qquad t = \tilde{\gamma} t' + \gamma vx'$$

- Substitute the inverse transformation for x into the transformation for x' to get:

$$x' = x'(\gamma^2 - \tilde{\gamma}\gamma v^2) + t'(\gamma\tilde{\gamma}v - \tilde{\gamma}^2 v)$$

  which must be true for all x', t', so the terms in the second parentheses must be zero: $\gamma = \tilde{\gamma}$, and the terms in the first parentheses must be 1:

$$\gamma = \tilde{\gamma} = \frac{1}{\sqrt{1 - v^2}}.$$

- Finally, we argue that the transverse coordinates (y, z) do not change between frames:

$$y' = y$$
$$z' = z$$

  If they did, there could be a contraction or elongation of space along those dimensions. But any such change of transverse coordinates would violate the principle of relativity, as illustrated in Fig. 8.

The Lorentz transforms from S to S' and back

From these arguments you can construct the following results for the $S \to S'$ transformation

$$t' = \gamma(t - vx) \tag{1a}$$
$$x' = \gamma(x - vt) \tag{1b}$$
$$y' = y \tag{1c}$$
$$z' = z \tag{1d}$$

with $\gamma = 1/\sqrt{1 - v^2}$. The $S' \to S$ transformation is given by $v \to -v$

$$t = \gamma(t' + vx') \tag{2a}$$
$$x = \gamma(x' + vt') \tag{2b}$$
$$y = y' \tag{2c}$$
$$z = z' \tag{2d}$$

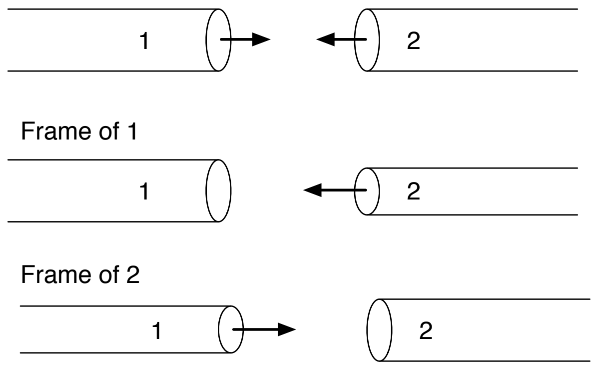

Figure 8: Consider two rods moving towards one another along the x axis, with transverse circular profiles as shown. In the rest frame of rod 1, rod 2 might have a contraction of its transverse coordinates, leaving a smaller indentation in rod 1's profile. In the rest frame of rod 2, rod 1 might have a contraction of its transverse coordinates, leaving a smaller indentation in rod 2's profile. By the Principle of Relativity of inertial frames, in this symmetric case, neither can happen; there can be no change of transverse coordinates.

From these equations you can derive the important invariant

$$t^2 - x^2 - y^2 - z^2 = t'^2 - x'^2 - y'^2 - z'^2 \equiv \tau^2, \tag{3}$$

where τ is called the *proper time*. This invariant applies to the *interval between two events*; the first event in S is at $t = x = y = z = 0$ and the same for S', since we've set up our standard configuration such that the origin event coincides in both frames (if it helps, but Δ's in front of all terms to remind you that it's an interval between two events). Then, τ is the time as measured by an observer at rest in an inertial frame (say, S) in which the two events are both at $x = y = z = 0$.

You may be bothered about the apparent dimensionality dissonance; if so, put in the c's:

$$t' = \gamma(t - vx/c^2) \tag{4a}$$

$$\gamma = 1/\sqrt{1 - (v/c)^2} \tag{4b}$$

$$(ct)^2 - x^2 - y^2 - z^2 = (ct')^2 - x'^2 - y'^2 - z'^2 \tag{4c}$$

and so on.

Relativity of simultaneity

To use the Lorentz transformation, first describe the physical phenomenon of interest in terms of events, then relate the coordinates of the events using these equations. The key to resolving many apparent paradoxes is that spatially separated events that are simultaneous (occur at the same time) in one frame of reference may not be simultaneous in other frames.

Clocks and rulers

Let us look at the Lorentz transformation in terms of the clocks and rulers notionally used to set up the coordinates. This introduces the ideas of *time dilation* and *length contraction*.

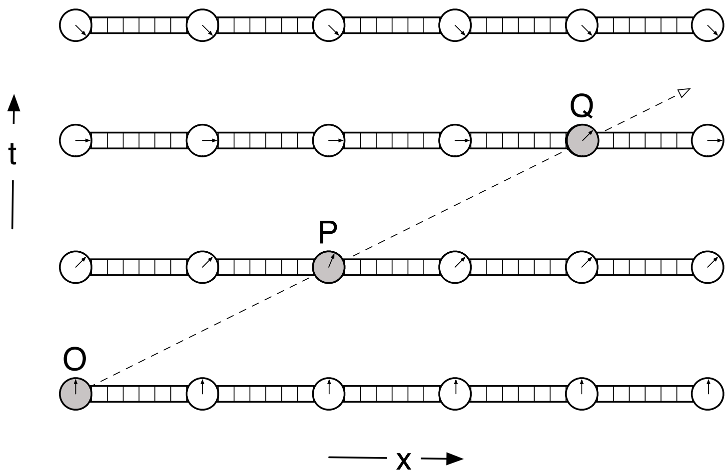

Figure 9: Clock at the origin of the S' frame (grey) moving through the S frame. The S frame clock at the position of the grey clock is not shown, but would, of course read the same time as the other clocks in the lattice.

I set up clocks and rulers to define coordinates in my S frame of reference. Let's suppose that you are in the inertial frame S'. You would do exactly the same thing, and set up a lattice of clocks and rulers defining the coordinates (x', y', z') and time t' of an event. We have agreed to origins for our space coordinates that coincide at some time (an event!), and set our clocks to zero at that event, so that $x = y = z = t = 0$ corresponds to $x' = y' = z' = t' = 0$. You and I agree we have set up equally good ways of measuring space and time, since we did it in the same way, and of course the mechanics of the clocks are the same in our two inertial frames by the principle of relativity. However we will not necessarily agree on the coordinates assigned to events.

Time dilation

For example consider two events: $\mathcal{O}$ the coincidence of the origins that has coordinates $x = y = z = t = 0$ and $x' = y' = z' = t' = 0$ (we agree on this one), and $\mathcal{P}$ that occurs at a later time, and at the spatial origin of in your frame $x' = y' = z' = 0$. Of course the spatial coordinates in my frame are different, just as they are in Galilean physics, since your origin is moving in my frame. But in addition the times are different: in my S frame, the clock at $x' = 0$ in the S' frame is at $x = vt$; from Eqn. 1a, the time on your clock appears to read $t' = \gamma(1 - v^2)t = \sqrt{1 - v^2}t < t$, or slower than my clock. That is, the two clocks present at the event $\mathcal{P}$, one of your clocks and one of mine, give *different* readings.

How can that be? How can one clock read earlier than the other, since there is a "symmetry" between the frames — you are moving relative to me, and I am moving relative to you? Which one reads the shorter time? The answer is that there is no symmetry for the events described: you have a single clock present at both events; I have different, but synchronized, clocks that read the times of $\mathcal{O}$ and $\mathcal{P}$. Your clock is said to read the *proper time* interval between the two events. It is a *shorter* time than I read from my clocks. This is shown in the S frame for three events $\mathcal{O}$, $\mathcal{P}$, $\mathcal{Q}$ (the clock at the S' origin at three times) in Fig. 9. This is the phenomenon of time dilation. Quantitatively, the event $\mathcal{P}$ has the coordinates $t', x' = 0$, and using Eq. (2a) gives $t = \gamma t'$. Remember $\gamma > 1$, so that $t > t'$.

Of couse, this is usually a small effect for macroscopic objects; for example, a jet plane traveling at 300 m/s (or 671 miles/hour) has $\gamma = 1/\sqrt{1-(v/c)^2}$ is greater than 1 by about 0.5×10^{-12}. But (as we'll see next week) a proton in the Large Hadron Collider with energy $E = 7$ TeV has $\gamma \approx 7,000$, so that its velocity is $v = (1 - 1 \times 10^{-8})c$; a highly relativistic system.

Length contraction

Another example is the measurement of the length of a moving object. To measure the length of an S' ruler in frame S we find the coordinate x at $t = 0$ for coordinate $x' = L$. Remember $x = 0, t = 0$ implies $x' = 0, t' = 0$ so that we are determining the coordinates in S of the two ends of the S' ruler at the same time $t = 0$ — which is how we measure a length! Using Eq. (1b) gives $x = L/\gamma$: we measure the moving ruler to be length contracted.

Velocity transformation or addition

A particle moves at constant velocity $\vec{u} = d(x, y, z)/dt$ in S frame, as uniform motion between $(0, 0, 0)$ at $t = 0$ to (x, y, z) at time t. The velocity $\vec{u}' = d(x', y', z')/dt'$ in the S' frame is:

$$
\begin{aligned}
u'_x &= \frac{x'}{t'} = \frac{\gamma(x - vt)}{\gamma(t - vx)} = \frac{u_x - v}{1 - u_x v} \\
u'_y &= \frac{y'}{t'} = \frac{y}{\gamma(t - vx)} = \frac{u_y}{\gamma(1 - u_x v)}, \qquad \gamma \equiv \gamma_v = \frac{1}{\sqrt{1 - v^2}} \\
u'_z &= \frac{z'}{t'} = \frac{z}{\gamma(t - vx)} = \frac{u_z}{\gamma(1 - u_x v)}
\end{aligned}
$$

The inverse transformation of velocities:

$$
u_x = \frac{u'_x + v}{1 + u'_x v}, \quad u_y = \frac{u'_y}{\gamma(1 + u'_x v)}, \quad u_z = \frac{u'_z}{\gamma(1 + u'_x v)}
$$