Physics 106a, Caltech — 12 November, 2019

Lecture 13: Central Forces – Scattering States

In this lecture we discuss scattering problems, particularly *Rutherford scattering* in a 1/r potential.
First, some review.

Unbound Orbits in Central Potentials: Scattering

- Review of planetary orbits
- Repulsive $1/r$ potential: Rutherford scattering
- Scattering problems: what do we want to know?
- Particle trajectory for repulsive $1/r$ potential
- Rutherford scattering cross section
- Effect of finite target mass: kinematics and dynamics

Review of planetary orbits — Rotational symmetry

- Eliminate center of mass motion
- Angular momentum $\vec{l}$ is conserved
- Orbit lies in plane perpendicular to $\vec{l}$: specify by (r, ϕ)
- Lagrangian for 2d motion
$$\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - V(r)$$
- Constant of the motion
$$l = \mu r^2 \dot{\phi}$$
- Kepler's second law: constant rate of sweeping out area; $\dot{A} = \frac{1}{2}r^2\dot{\phi} = l/2\mu = \text{constant}$.

Scattering Problems — Repulsive k/r potential

- Hamiltonian is total energy and is constant
$$E = \frac{1}{2}\mu\dot{r}^2 + V_{\text{eff}}(r) \quad \text{with} \quad V_{\text{eff}}(r) = \frac{l^2}{2\mu r^2} + V(r)$$
- For $V(r) = \frac{k}{r}$ (with $k > 0$), introduce $u = \frac{1}{r}$ so that $\dot{r} = \dot{\phi}\frac{dr}{d\phi} = -\frac{l}{\mu}\frac{du}{d\phi}$
$$E = \frac{l^2}{2\mu}\left[\left(\frac{du}{d\phi}\right)^2 + \left(u + \frac{1}{p}\right)^2 - \frac{1}{p^2}\right] \quad \text{with } p = \frac{l^2}{\mu k} > 0$$
- $u(\phi)$ is sinusoidal about $-1/p$
$$\frac{1}{r} = -\frac{1}{p} + \frac{\epsilon}{p}\cos\phi$$
Physical solutions $r > 0$ for $\cos\phi > 1/\epsilon$
- Conic section with eccentricity $\epsilon > 1$ (hyperbola) and $E = \frac{l^2}{2\mu p^2}(\epsilon^2 - 1)$

Hyperbolic orbits: $\epsilon > 1$

These exist for attractive potentials $V(r) = -k/r$, and also for repulsive ones $V(r) = k/r$. Note in both cases I take k to be positive and I define $p = l^2/\mu k > 0$. The equations are

$$\begin{array}{lll} \text{Attractive, } k>0 & \frac{1}{r} = \frac{1}{p}\left[1+\epsilon\cos\phi\right] & \text{for } \cos\phi > -\frac{1}{\epsilon} \\ \text{Repulsive, } k<0 & \frac{1}{r} = \frac{1}{p}\left[-1+\epsilon\cos\phi\right] & \text{for } \cos\phi > \frac{1}{\epsilon} \end{array} \tag{1}$$

with $\epsilon > 1, E > 0$ (solutions for the repulsive case only exist for this range). Physical solutions $r > 0$ only exist for the range of angles specified. The geometry of the orbits is shown in Fig. 1 (using the same value of k in the two cases). We see that for attractive potentials, the orbit passes *behind* the focus (the sun), while for repulsive potentials the orbit passes in front of the focus.

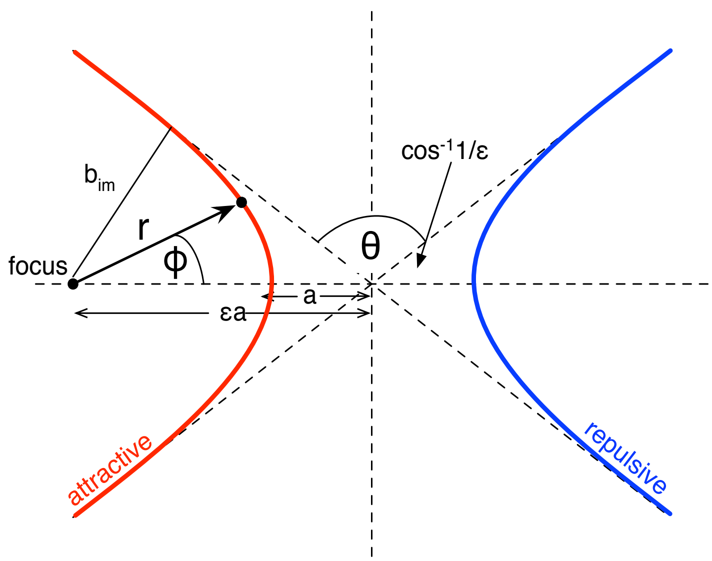

Figure 1: Hyperbolic orbits for attractive and repulsive $1/r$ potentials.

Rutherford scattering

The hyperbolic orbits for the repulsive case allow us to investigate the scattering of alpha particles (charge $2|e|$) off a nucleus (charge $Z|e|$). We will suppose the target nucleus is heavy, so that its recoil is negligible and we can consider scattering off a fixed potential. In a scattering problem we have a parallel beam of incident particles all with the same energy. The scattering angle θ is determined by how close the alpha particle comes to the nucleus, which depends on the *impact parameter* b_{im} for that particle (see Fig. 2). We are usually interested in the rate of scattering of particles into a scattering angle between θ and $\theta + d\theta$, which corresponds to a solid angle $d\Omega = 2\pi \sin\theta d\theta$. These particles are ones with an impact parameter between b_{im} and $b_{\text{im}} + db_{\text{im}}$, i.e. the particles hitting an area $d\sigma = 2\pi b_{\text{im}} db_{\text{im}}$ (see Fig. 3).

Rutherford Scattering Experiment

Scattering of alpha particles off thin gold film [Geiger and Marsden, 1909].

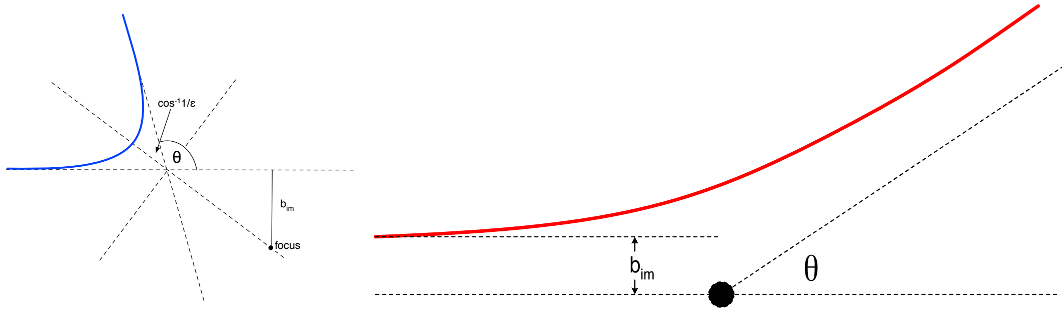

Figure 2: Two different views of the scattering geometry, showing the impact parameter $b_{\rm im}$
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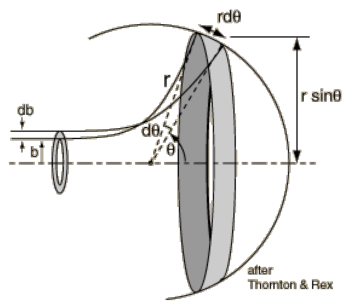


Figure 3: Relationship between the cross-sectional annulus (between b and $b+db$) and the scattering angle.

The experimenters set up a beam of relatively light nuclei: alpha (α) particles, which are helium nuclei, with mass $A_{He}m_N$, $A_{He} = 4$ and $m_n = 1/N_A$ is the mass of a nucleon (proton or neutron, almost the same). Avagadro's number is $N_A = 6 \times 10^{23}$ nucleons per gram.

The target is a thin gold foil (thickness = ℓ), composed of nuclei of mass $A_{Au}m_p$, $A_{Au} = 79$. Gold has a (macroscopic, well known) density of $\rho = 19.32\, g/cm^3$. They made the foil very thin in the hope that a beam particle would typically scatter off of only one nucleus; else, it is *multiple scattering*, requiring statistical analysis.

The beam "sees" an array of roughly spherical target nuclei that appear in "cross section", as circles. If the target is thin enough, the circles don't overlap (no multiple scattering). The *geometric cross section* is $\sigma_g = \pi R_N^2$, where the nuclear radius $R_N = A_{Au}^{1/3} R_n$, where $R_n = 1 \times 10^{-13}$ cm (1 fm) is the radius of a nucleon (of course, Rutherford, Geiger and Marsden didn't know that; the point of their experiment was to measure the microscopic cross-section of the nucleus). In nuclear physics, this is measured in units of *barns*, where 1 barn is 10^{-24} cm.

As we saw above, the cross-section really depends on scattering angle; it's not a "black disk" (hit or miss). Because it is microscopic, it must be properly computed in quantum mechanics (you'll do that in Ph 125), and of course, Rutherford, Geiger and Marsden didn't know that either. So the geometric cross-section is just an order-of-magnitude estimate of the total cross-section $\sigma = \int (d\sigma/d\Omega) d\Omega$.

The beam is characterized by a flux $J = \dot{N}_{beam}$: a number of beam α particles per unit time. Assuming we're not in the multiple scattering regime, the probability of a beam particle hitting a target nucleus is the cross-sectional area of the target σ divided by the area per target nucleus $A_N m_n/(\rho\ell)$ (again, ℓ is the thickness of the target). After a time t, $N_{beam} = Jt$ beam particles have impinged, and

$$N_{scat} = N_{beam}\frac{\rho\ell}{A_N m_n}\sigma \tag{2}$$

have scattered (verify this!). Everything in this equation is known (N_{beam}, ρ, ℓ, A, m_n), or measured (N_{scat}), except the microscopic nuclear cross-section σ, which is computed from this formula.

The number of scattered beam particles is measured as a function of scattering angle, $N_{scat}(\theta) = dN_{scat}/d\theta$ (θ must be measurably greater than zero in order for the beam particle to be considered as having scattered). From this we compute the measured microscopic differential cross-section $d\sigma^{meas}/d\Omega$.

The differential scattering cross-section

The scattering is described by the *differential scattering cross-section*

$$\frac{d\sigma}{d\Omega} = \left|\frac{2\pi b_{\rm im} db_{\rm im}}{2\pi\sin\theta d\theta}\right| = \frac{b_{\rm im}}{\sin\theta}\left|\frac{db_{\rm im}}{d\theta}\right| \tag{3}$$

(we put the mod in since $d\sigma/d\Omega$ is defined to be postive) so that all we need to know from the hyperbolic orbit calculation is $\theta(b_{\rm im})$. This is the topic of Hand and Finch problem 4-28. Here is the solution.

We use the results for scattering off a 1/r potential. The trick is to relate the parameters of the orbit calculation, in particular the energy E and angular momentum l, to those of the scattering problem. Fig. 2:

$$E = \tfrac{1}{2}\mu v_\infty^2, \qquad l = \mu v_\infty b_{\rm im}, \tag{4}$$

with v_∞ the speed of the particles in the incident beam. Now use Eqs. (1) for the repulsive case with $p = l^2/\mu k$, $k = Z_{Au}Z_\alpha e^2$. Also, from Lecture 8, the energy is given in terms of the angular momentum l and the eccentricity ϵ by

$$E = \frac{l^2}{2\mu p^2}(\epsilon^2 - 1) = \frac{1}{2}\frac{k^2}{l^2/\mu}(\epsilon^2 - 1) \tag{5}$$

where the second expression is given by substituting for p. The scattering angle θ is given in terms of ϵ by

$$\theta = \pi - 2\phi_\infty \quad \text{where} \quad \phi_\infty = \phi(r \to \infty) = \cos^{-1}\left(\frac{1}{\epsilon}\right) \tag{6}$$

again using Eq. (1) (see also Fig. 1). This gives $\sin(\theta/2) = 1/\epsilon$, so that

$$E = \frac{1}{2}\frac{k^2}{l^2/\mu}[\mathrm{cosec}^2(\theta/2) - 1] = \frac{1}{2}\frac{k^2}{l^2/\mu}\cot^2(\theta/2). \tag{7}$$

Equations (4) give

$$l^2/\mu = \mu v_\infty^2 b_{\rm im}^2 = 2Eb_{\rm im}^2. \tag{8}$$

Substituting into Eq. (7), rearranging, and taking the square root gives

$$b_{\rm im} = \frac{|k|}{2E}\cot\left(\frac{\theta}{2}\right), \tag{9}$$

the relationship $b_{\rm im}(\theta)$ we need. Using Eq. (3) gives the Rutherford differential scattering cross-section

$$\frac{d\sigma}{d\Omega} = \frac{b_{\rm im}}{\sin\theta}\left|\frac{db_{\rm im}}{d\theta}\right| = \left(\frac{k}{4E}\right)^2\frac{1}{\sin^4\frac{\theta}{2}}. \tag{10}$$

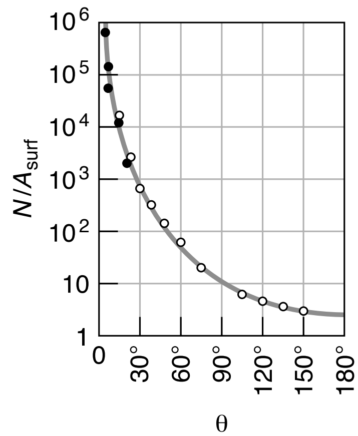
Experimental Results - measuring the cross section

The experimental results came as a shock to Rutherford:

> It was quite the most incredible event that has ever happened to me in my life. It was almost as incredible as if you fired a 15-inch shell at a piece of tissue paper and it came back and hit you.
>
> On consideration, I realized that this scattering backward must be the result of a single collision, and when I made calculations I saw that it was impossible to get anything of that order of magnitude unless you took a system in which the greater part of the mass of the atom was concentrated in a minute nucleus. It was then that I had the idea of an atom with a minute massive center, carrying a charge.

[Ernest Rutherford]

We can compare the measured and theoretical cross-sections, varying some unknown parameter until one gets a match. In the case of Geiger & Marsden, they considered one free parameter, R_N (the radius of the target nucleus). (The radius of the beam nucleus is approximated to be negligible; or, one is measuring the *joint* beam-target cross-section.)



From *The Atomic Nucleus* by R. D. Evans

The closest distance of approach was 30 fm (3×10^{-14}m) for 7.78 MeV alpha particle at 150 deg scattering. So they concluded that $R_N < 30$ fm, or $R_n < 30/(79)^{1/3} = 7$ fm, which was *much* smaller than the distance between atoms in gold. Hence, Rutherford's great surprise. (As mentioned above, we now know $R_n = 1$ fm.)

Finite mass target particle

If the target particle is not infinitely heavy, it will recoil in the scattering, and the scattering problem is more complicated: for example the energy and speed of the outgoing scattered particle (in the lab frame) will not be the same as the incoming values. The scattering problem separates into two parts: the *dynamics* – the probability of scattering at some angle which depends on solving for the particle trajectories in the interaction potential as we have just done; and the *kinematics* – how the outgoing energy depends on the scattering angle, what is the momentum of the outgoing target particle etc., which are determined simply by conservation of energy and momentum. The scattering with a finite mass target is most easily addressed by transforming to the center of mass frame, solving for the dynamics there, and then transforming back to the original "laboratory" frame.

Kinematics

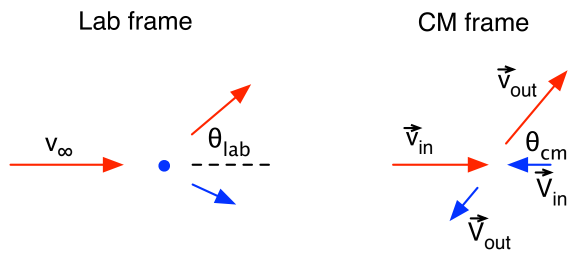


Figure 4: Scattering geometry in the lab and center of mass frames. Only the incoming and outgoing velocity vectors (not the details of the trajectory etc.) are depicted. In the center of mass frame the incoming velocities are colinear, as are the outgoing velocities. Also the outgoing speed is the same as the incoming speed for that particle.

In the center of mass frame the scattering geometry is simple. Calling the velocity of the mass m particle $\vec{v}$ and that of the mass M particle $\vec{V}$ for elastic scattering $|\vec{v}_{\rm out}| = |\vec{v}_{\rm in}|$ and $|\vec{V}_{\rm out}| = |\vec{V}_{\rm in}|$. Also $m\vec{v}_{\rm in,out} = -M\vec{V}_{\rm in,out}$ since the total momentum is zero. If in the laboratory frame the incoming velocity of the mass m particles is $v_\infty \hat{x}$, the center of mass velocity is

$$\vec{V}_{\rm cm} = \frac{m}{M+m} v_\infty \hat{x}, \tag{11}$$

and the incoming velocity of the mass m in the center of mass frame is

$$\vec{v}_{\rm in}|_{\rm cm} = v_\infty \hat{x} - \vec{V}_{\rm cm} = \frac{M}{M+m} v_\infty \hat{x}. \tag{12}$$

Since the speed is the same after the scattering

$$\vec{v}_{\rm out}|_{\rm cm} = \frac{M}{M+m} v_\infty (\cos\theta_{\rm cm}\hat{x} + \sin\theta_{\rm cm}\hat{y}). \tag{13}$$

Transforming back to the laboratory frame by adding $\vec{V}_{\rm cm}$ to all the velocities

$$\vec{v}_{\rm out}|_{\rm lab} = \frac{M}{M+m} v_\infty [(\cos\theta_{\rm cm} + m/M)\hat{x} + \sin\theta_{\rm cm}\hat{y}]. \tag{14}$$

This gives the scattering angle $\theta_{\rm lab}$ in the laboratory frame

$$\tan\theta_{\rm lab} = \frac{\sin\theta_{\rm cm}}{\cos\theta_{\rm cm} + m/M}. \tag{15}$$

Dynamics

Now I calculate the scattering dynamics in the center of mass frame. The particle trajectories in the center of mass frame are given by

$$\vec{r}_m(t) = \frac{M}{M+m}\vec{r}(t), \quad \vec{r}_M(t) = -\frac{m}{M+m}\vec{r}(t), \tag{16}$$

where $\vec{r} = \vec{r}_m - \vec{r}_M$ is the difference coordinate used in the orbit calculation. These are just scaled (and flipped in the later case) versions of the hyperbola traced out by $\vec{r}(t)$, as shown in Fig. 5. This

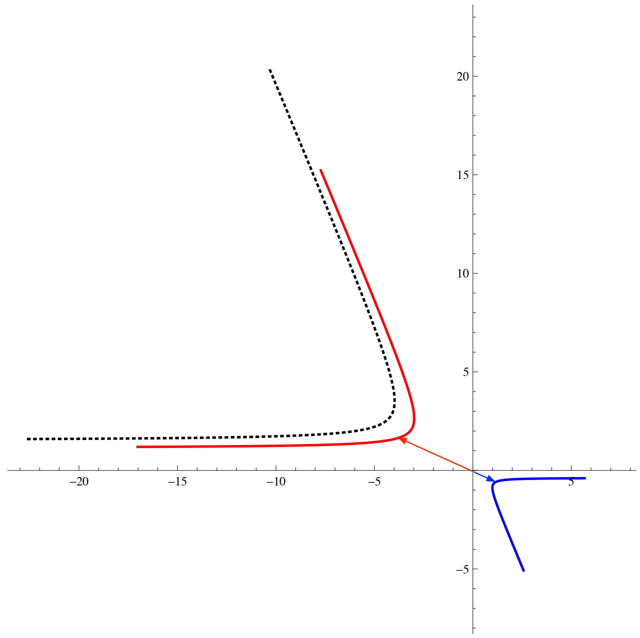


Figure 5: Scattering hyperbolas for a particle of mass m off one with mass M with $M/m = 3$: the dashed curve is the hyperbola traced out by the difference coordinate $\vec{r} = \vec{r}_m - \vec{r}_M$ given by the orbit calculation. I used $p = 1$ and an eccentricity $\epsilon = 1.2$ which fixes the scattering angle as 146°. The red curve is the trajectory of particle m given by $\vec{r}_m(t)$ and the blue curve is the trajectory of particle M given by $\vec{r}_M(t)$, both in the center of mass frame. The foci of the hyperbolas are all at the origin. The curves are plotted over the same time interval, and the heavy mass moves a smaller distance than the light mass. The red and blue arrows are $\vec{r}_m, \vec{r}_M$ at some particular time.

means that the scattering angle $\theta_{\rm cm}$ of particle m in the center of mass frame is the same as the scattering angle of the reduced mass μ scattering off a stationary center calculated in the previous section.

To calculate the differential scattering cross-section in the laboratory frame from the differential scattering cross section calculated for the reduced mass particle in the previous section, use the fact that the incoming fluxes are the same (both incoming speeds are $\vec{v}_\infty$) and $2\pi \sin\theta \, d\sigma/d\Omega \, d\theta$ counts the same outcoming particles when evaluated in the two frames. Thus

$$\left.\frac{d\sigma}{d\Omega}\right|_{\rm lab} = \frac{\sin\theta_{\rm cm}}{\sin\theta_{\rm lab}} \frac{d\theta_{\rm cm}}{d\theta_{\rm lab}} \left.\frac{d\sigma}{d\Omega}\right|_{\rm cm}, \tag{17}$$

with $\theta_{\rm cm}, \theta_{\rm lab}$ related by Eq. (15).

See the discussion in §3.11 of Goldstein, Poole and Safko for more details.