Physics 106a, Caltech — 5, 7 November, 2019

Lectures 11 & 12: Central Forces – Bound States

Today we discuss the *Kepler problem* of the orbital motion of planets and other objects in the gravitational field of the sun.

Calculating the motion of a planet in the gravitational potential of the Sun illustrates many important ideas in Lagrangian mechanics:

- Symmetries $\rightarrow$ conserved quantities – Noether's theorem
- Use conserved quantities to reduce number of variables and the equation of motion (EOM)
  - Planetary dynamics: reduce from 6 to 1!
- Constant Hamiltonian (first integral of Euler-Lagrange equation), conservation of energy
- Solution of one-dimensional problems
  - qualitative – motion of particle in effective potential
  - method of quadratures – reduce to integral
- Particular example of 1/r potential

We begin by exploiting the *homogeneity* and *isotropy* of space, because these *symmetries* lead to *conservation laws* that will greatly simply the problem.

Ignorable coordinates

If a Lagrangian does not depend on a particular coordinate q_k, so that $\partial L/\partial q_k = 0$, then by the Euler-Lagrange equations,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = \frac{\partial L}{\partial q_k} = 0, \tag{1}$$

so that $\partial L/\partial \dot{q}_k$ is a constant of motion, or conserved quantity. Of course, if $\partial L/\partial \dot{q}_k = 0$ as well, then the Lagrangian doesn't depend on either q_k or $\dot{q}_k$; it isn't even a relevant coordinate to the problem.

Noether's Theorem

This formalizes the idea that a continuous or differential symmetry leads to conserved quantities (constants of the motion).

We parameterize the symmetry operation by describing how the generalized coordinates q_k change with a continuous parameter s (e.g. this would be a displacement, for translational symmetry, or a rotation angle for rotational symmetry). Introduce $Q_k(s)$ which gives the different versions of q_k as the symmetry operation is performed. We choose the definition so that $Q_k(s=0) = q_k$. The transformation represented by s is a *symmetry operation* if the Lagrangian is unchanged:

$$\frac{d}{ds}L(\{Q_k\}, \{\dot{Q}_k\}, t) = 0. \tag{2}$$

Expressing the total derivative in terms of the partials

$$\frac{dL}{ds} = \sum_k \left[\frac{\partial L}{\partial Q_k} \frac{\partial Q_k}{\partial s} + \frac{\partial L}{\partial \dot{Q}_k} \frac{\partial \dot{Q}_k}{\partial s} \right] \tag{3}$$

$$= \sum_k \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{Q}_k} \right) \frac{\partial Q_k}{\partial s} + \frac{\partial L}{\partial \dot{Q}_k} \frac{\partial \dot{Q}_k}{\partial s} \right] \tag{4}$$

$$= \frac{d}{dt} \left[\sum_k \left(\frac{\partial L}{\partial \dot{Q}_k} \right) \frac{\partial Q_k}{\partial s} \right] = \frac{d}{dt} \sum_k p_k \frac{\partial Q_k}{\partial s}, \tag{5}$$

where we used the Euler-Lagrange equation(s) to replace $\partial L/\partial \dot{Q}_k$ with $d/dt(\partial L/\partial \dot{Q}_k)$, and we make use of the conjugate momentum $p_k \equiv \partial L/\partial \dot{Q}_k$ evaluated at $s = 0$.

If such a transformation is a symmetry, then $dL/ds = 0$. This gives us the conserved quantity (which we choose to evaluate at $s = 0$)

$$I(\{q_k\}, \{\dot{q}_k\}, t) = \sum_k p_k \left. (\partial Q_k/\partial s) \right|_{s=0}. \tag{6}$$

Note that the Euler-Lagrange equation is used to get the second line Eq. (4): it is not correct to say "symmetry implies a conserved quantity"; rather symmetry *together with Lagrangian equations of motion* imply a conserved quantity. For example, adding dissipation (which is difficult to incorporate into the Lagrangian formalism) usually destroys conservations laws, e.g. momentum.

Example 1: Translational symmetry for M particles

Define the extended coordinates $\vec{R}_i(\vec{d}) = \vec{r}_i + \vec{d}$ with $\vec{d}$ a translation. So here, we have three quantities that, in turn, play the role of s: d_x, d_y, d_z. For a Lagrangian that is translationally invariant for displacements in the x-direction the conserved quantity is

$$P_x = \sum_i \vec{p}_i \cdot (\partial \vec{R}_i/\partial d_x)_{d=0} = \sum_i p_{ix}, \tag{7}$$

which is the total x-momentum. For full translational invariance, the vector total momentum $\vec{P}$ is conserved. Of course, if there is an external potential $V(\vec{r})$ that varies with position, there is no translational invariance, and no conservation of momentum.

Example 2: Rotations about the z-axis

For Cartesian coordinates (x_i, y_i) of the particles we introduce the rotated coordinates depending on the rotation angle ϕ about the z-axis (i.e. $s \equiv \phi$ in this case)

$$X_i(\phi) = x_i \cos\phi - y_i \sin\phi, \tag{8}$$

$$Y_i(\phi) = x_i \sin\phi + y_i \cos\phi. \tag{9}$$

If the Lagrangian is unchanged under rotation about the z-axis gives us the conserved quantity

$$L_z = \sum_i p_{xi} \left. \frac{\partial X_i}{\partial \phi} \right|_{\phi=0} + p_{yi} \left. \frac{\partial Y_i}{\partial \phi} \right|_{\phi=0} \tag{10}$$

$$= \sum_i p_{xi}(-y_i) + p_{yi} x_i = \sum_i (\vec{r}_i \times \vec{p}_i)_z. \tag{11}$$

Thus rotational symmetry about the z-axis implies the conservation of the z-component of the angular momentum. Full rotational symmetry, about any axis, implies the conservation of the vector angular momentum $\vec{L}$.

Central forces

We first consider the more general problem of two point particles of mass M_1, M_2 interacting with a *central force* (one directed between the points or the centers of the spheres). There are 6 degrees of freedom, e.g. the position vectors $\vec{r}_1, \vec{r}_2$ described by the Lagrangian

$$L = \frac{1}{2} M_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} M_2 \dot{\vec{r}}_2^{\,2} - V(|\vec{r}_1 - \vec{r}_2|) \tag{12}$$

with $V(r)$ giving the central force. The symmetries and consequent conservation laws allow us to reduce the problem to solving for 1 degree of freedom — an enormous simplification!

Translational symmetry: Translational symmetry gives the conservation of total momentum $\vec{P} = \vec{p}_1 + \vec{p}_2 = M_1 \dot{\vec{r}}_1 + M_2 \dot{\vec{r}}_2$ (using $\vec{p}_1 = \partial L / \partial \dot{\vec{r}}_1$ etc.). This can be shown using Noether's theorem (see above) or directly from the equations of motion. We therefore introduce the center of mass coordinate $\vec{R}_{\rm cm} = (M_1 \vec{r}_1 + M_2 \vec{r}_2)/(M_1 + M_2)$, so that $\vec{P} = (M_1 + M_2) \dot{\vec{R}}_{\rm cm}$. As the second coordinate we use the translationally invariant difference vector $\vec{r} = \vec{r}_1 - \vec{r}_2$. The Lagrangian becomes

$$L = \frac{1}{2} M \dot{\vec{R}}_{\rm cm}^{2} + \frac{1}{2} \mu \dot{\vec{r}}^{\,2} - V(r) \tag{13}$$

with $M = M_1 + M_2$ the total mass, and $\mu = M_1 M_2 / (M_1 + M_2)$ the reduced mass. Indeed we see that $\vec{R}_{\rm cm}$ is ignorable ($\partial L / \partial \vec{R}_{\rm cm} = 0$), and $\vec{P}$ is the conserved conjugate momentum $\vec{P} = \partial L / \partial \dot{\vec{R}}_{\rm cm}$. The first term in L is constant, and thus we now only have to consider the relative motion described by the Lagrangian

$$L = \frac{1}{2} \mu \dot{\vec{r}}^{\,2} - V(r). \tag{14}$$

Rotational symmetry: The Lagrangian only involves scalars (vectors appear only in dot products with other vectors; and dot products are rotationally invariant), and is unchanged by any rotation of the whole system. This means the angular momentum $\vec{l} = (l_x, l_y, l_z)$ is a constant of the motion. Again this follows from Noether's theorem (see above). Alternatively, in spherical polar coordinates

$$L = \frac{1}{2} \mu (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) - V(r). \tag{15}$$

Now ϕ is ignorable, and the conjugate momentum $l_z = \partial L / \partial \dot{\phi} = \mu r^2 \sin^2 \theta \, \dot{\phi}$ is a constant of the motion. The direction of z is arbitrary, so this means the full vector $\vec{l}$ is constant.

Planar motion: For constant $\vec{l} = \vec{r}(t) \times \vec{p}(t)$ the vectors $\vec{r}(t)$ and $\vec{p}(t)$ always lies in the fixed plane perpendicular to $\vec{l}$. (This is not true in quantum mechanics, which is a wave theory, and waves are extended in space, not confinable to a plane. However, in QM, the angular solutions to the central force problem are spherical harmonics, which are standing waves in θ, and travelling waves in ϕ).

Choosing the z-axis along the direction of $\vec{l}$ means that the orbital plane is $\theta = \pi/2$. Also, $\dot{\theta} = 0$, and the Lagrangian (above) for motion in the plane reduces to

$$L = \frac{1}{2} \mu (\dot{r}^2 + r^2 \dot{\phi}^2) - V(r). \tag{16}$$

Again, ϕ is ignorable ($\partial L / \partial \phi = 0$), so $l = \partial L / \partial \dot{\phi} = \mu r^2 \dot{\phi}$ is constant. This immediately gives Keplers's second law, that the radius vector sweeps out area at a constant rate, since $\dot{A} = \frac{1}{2} r^2 \dot{\phi} = l / 2\mu$.

One dimensional reduction: Putting this Lagrangian into the Euler-Lagrange equation for r yields the EOM:

$$\mu\ddot{r} - \mu r\dot{\phi}^2 + dV/dr = 0 \tag{17}$$

and eliminating $\dot{\phi}$ in favor of the constant $l = \mu r^2\dot{\phi}$,

$$\mu\ddot{r} - \frac{l^2}{\mu r^3} + dV/dr = 0. \tag{18}$$

This second term is often called a *centrifugal force*; but as we know from Ph 1, centrifugal forces are not real forces; in this case, it is simply the consequence of rotational motion. Still, we can write it as the gradient of a thing that looks like a potential:

$$\mu\ddot{r} + dV_{\text{eff}}/dr = 0 \qquad \text{with} \qquad V_{\text{eff}}(r) = \frac{l^2}{2\mu r^2} + V(r). \tag{19}$$

This is a one dimensional problem in an effective potential with an additional term from the rotational motion; we shall call that term the *angular momentum potential barrier*.

Note that we eliminated $\dot{\phi}$ from the EOM using the constant l, *after* applying the Euler-Lagrange equations. You may *not* eliminate $\dot{\phi}$ from the Lagrangian – this gives the *wrong* answer (try it and see!). The Lagrangian must be expressed in terms of the velocities, *not* the momenta, and the Lagrangian coordinates $r, \dot{r}, \phi, \dot{\phi}$ must be treated as independently variable (not related by a constant l) in order for the Euler-Lagrange equations to work.

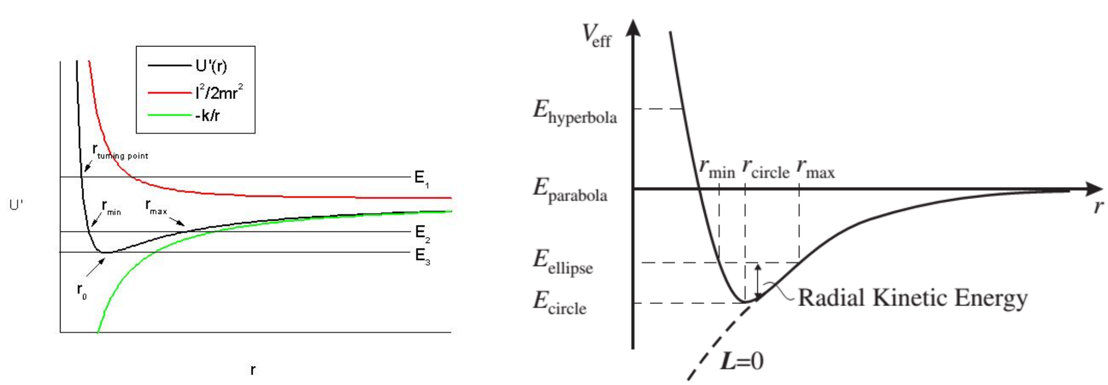
Constant H, E: To get the first integral of this equation we can use the fact that the Lagrangian is time independent ($\partial L/\partial t = 0$), so that the Hamiltonian H is constant, and since there are no time dependent constraints the Hamiltonian is equal to the total energy $H = E = T + V$ so that E is constant. Explicitly

$$E = \tfrac{1}{2}\mu\dot{r}^2 + V_{\text{eff}}(r) = \text{constant}. \tag{20}$$

Qualitative solution: Now specialize to the gravitational potential $V(r) = -k/r$ with $k = GM_1M_2 > 0$ so that

$$V_{\text{eff}}(r) = \frac{l^2}{2\mu r^2} - \frac{k}{r}. \tag{21}$$

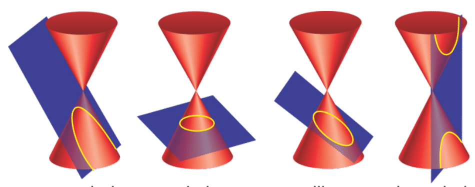
Sketching $V_{\text{eff}}(r)$ gives the qualitative solution: for $E < 0$ (set by initial conditions) the r motion oscillates over a finite range between $r_{\text{min}}, r_{\text{max}}$, so that the motion is bound, whereas for $E > 0$ the separation r increases to ∞ (maybe after one "bounce") so that the motion is unbound.



Plots are lifted from http://physics.stackexchange.com .

Note the following features:

- For $r \to \infty$, V_{eff} asymptotes to zero from below.
- For $r \to 0$, V_{eff} asymptotes to $+\infty$.
- In between, it has a minimum at $V_{\text{eff}}^{min} < 0$.
- For $E < V_{\text{eff}}^{min}$, there is no region where $E = (T + V) > V\text{eff}$, so no region where $T > 0$. Classically, we cannot have a negative kinetic energy (in quantum mechanics, we can; it's an evanescent quantum wave, and it is responsible for tunneling). So this is a *classically forbidden* region; no (classical) solution exists with that energy.
- for $E > V_{\text{eff}}^{min}$ but $E < 0$, we see that the motion is *bounded* between r_{min} and r_{max}. The regions $r < r_{min}$ and $r > r_{max}$ are *classically forbidden*, and r_{min} and r_{max} are the *classical turning points*. As we will see, the radial separation $r(\phi)$ oscillates between these two limits, in an ellipse.
- $E = V_{\text{eff}}^{min}$, we have $r_{min} = r_{max} = r_0$; the motion is circular.
- For $E \geq 0$, we have $T > 0$ even as $r \to \infty$; the motion is *unbounded*. For $E = 0$, the motion is a parabola, while for $E > 0$ it is an hyperbola.
  THE MIDTERM WILL COVER THE MATERIAL UP TO HERE, NOT WHAT FOLLOWS.
- All of these orbital shapes (elliptical, circular, parabolic and hyperbolic) are *conic sections*.



- So long as there is some angular momentum, $|l| > 0$, and for any finite energy E, we always have $r > r_{min}$. This is the *angular momentum barrier*, sometimes called the *centrifugal barrier*. It means that the relative separation can't fall to zero (unless $l = 0$, ie, a head-on collision). Of course, if the bodies have finite size, they can still merge together.

Solution for $r(t)$: Equation (20) gives us $\dot{r}$,

$$\dot{r} = \frac{dr}{dt} = \sqrt{\frac{2}{\mu}\left(E + \frac{k}{r} - \frac{l^2}{2\mu r^2}\right)} . \tag{22}$$

which we can formally integrate to get $t(r)$ starting from r_0 at $t = 0$

$$t = \sqrt{\frac{\mu}{2}} \int_{r_0}^{r} \frac{dr'}{\sqrt{E + \frac{k}{r'} - \frac{l^2}{2\mu r'^2}}} , \tag{23}$$

which then implicitly gives $r(t)$. See Hand and Finch pp148-9 for how to do this integral. This will give us $t(r)$ (once we have specified the constants $E < 0$ and $l \geq 0$). This can then (in principle) be inverted to give $r(t)$. Mathematicians call this approach to a solution "the method of quadratures".

One can do the same for $t(\phi)$ and $\phi(t)$. However, it's easier to get $\dot{\phi}(t)$ from $r(t)$ using $l = \mu r^2 \dot{\phi}$, then integrating to get $\phi(t)$.

Period of radial motion for a bound orbit: The period τ_r for the radial motion of a bound orbit is given by integrating from $r_{\rm min}$ to $r_{\rm max}$ and back

$$\tau_r = \sqrt{2\mu}\int_{r_{\rm min}}^{r_{\rm max}} \frac{dr'}{\sqrt{E + \frac{k}{r'} - \frac{l^2}{2\mu r'^2}}}\,. \tag{24}$$

Explicit solution: It is actually easier to find the shape of the orbit $r(\phi)$ rather than the trajectory $r(t)$. Use $\dot r = \dot\phi\, dr/d\phi$, and define $u \equiv 1/r$ (this trick only works for $1/r$ potentials; you will do this in solving the hydrogen atom in quantum mechanics). With these substitutions, write:

$$\dot r^2 = \dot\phi^2\left(\frac{dr}{d\phi}\right)^2 = \frac{l^2}{\mu^2}\left(\frac{1}{r^2}\frac{dr}{d\phi}\right)^2 = \frac{l^2}{\mu^2}\left(\frac{du}{d\phi}\right)^2 \qquad \text{with } u = 1/r\,. \tag{25}$$

Then a little algebra shows

$$E = \frac{l^2}{2\mu}\left[\left(\frac{du}{d\phi}\right)^2 + \left(u - \frac{1}{p}\right)^2 - \frac{1}{p^2}\right] \qquad \text{with } p = \frac{l^2}{\mu k}. \tag{26}$$

(This p is *not* momentum; it's yet another variable.)

This is the energy for simple harmonic motion $u(\phi)$ centered on $u = 1/p$. If you like, you can define $w \equiv u - 1/p$ to make an expression for the energy that looks explicitly like a SHO. You can also use $dE/d\phi = 0$ (since E is a constant) to turn this energy equation into an equation of motion for $u(\phi)$, which again, will look explicitly like a SHO:

$$\frac{d^2u}{d\phi^2} + u = \frac{1}{p} = \text{constant}. \tag{27}$$

The period of $u(\phi)$ is 2π, so that the bound orbits are *closed*. This is true for r^{-1} and r^2 potentials (Bertrand's theorem). The explicit solution is immediately obtained from the above SHO equation of motion:

$$u = \frac{1}{r} = \frac{1}{p} + \frac{\epsilon}{p}\cos\phi \tag{28}$$

with an arbitrary amplitude of the simple harmonic motion (set by the free parameter ϵ, which may be chosen positive). The free additive phase has been set to zero by a choice of the origin of ϕ. We will see that this describes *conic sections*, with ϵ the *eccentricity* parameter. Plugging Eq. (28) into Eq. (26) gives the energy for this orbit

$$E = \frac{l^2}{2\mu p^2}(\epsilon^2 - 1) = \frac{\mu k^2}{2l^2}(\epsilon^2 - 1)\,. \tag{29}$$

Shape of orbit: Equation (28) (or $r = p - \epsilon r\cos\phi$) is the equation in polar coordinates r, ϕ for a conic section with focus at the origin $r = 0$ and ϵ the *eccentricity.* For $\epsilon < 1$ the orbit is an ellipse, for $\epsilon > 1$ a hyperbola, and $\epsilon = 1$ a parabola. You can see this from geometrical definitions of conic sections, or by changing to Cartesian coordinates $r = \sqrt{x^2 + y^2}$, $r\cos\phi = x$ which gives

$$(1-\epsilon^2)x^2 + 2\epsilon p x + y^2 - p^2 = 0 \quad \text{with} \quad p = r + \epsilon r\cos\phi = \sqrt{x^2+y^2} + \epsilon x\,. \tag{30}$$

Rewriting,

$$\frac{(x-x_c)^2}{a^2} \pm \frac{y^2}{b^2} = 1 \tag{31}$$

with the + for $\epsilon < 1$ and the $-$ for $\epsilon > 1$, and

$$a = \frac{p}{|1-\epsilon^2|}, \qquad b = \frac{p}{\sqrt{|1-\epsilon^2|}}, \qquad x_c = -\frac{\epsilon p}{1-\epsilon^2}\,. \tag{32}$$

For $\epsilon = 0$, this is the equation for a circle of radius p. For $0 < \epsilon < 1$, this is an ellipse, with a the *semimajor axis* (along the x axis) length and b the *semiminor axis* (along y) length. The origin of the Cartesian coordinate system is the focus, displaced by $x_c = \epsilon a$ from the center of the ellipse.

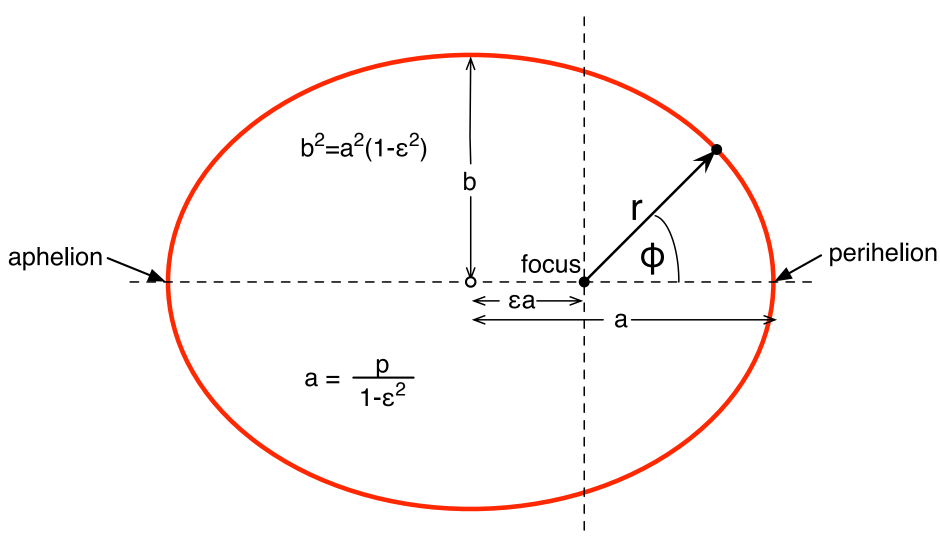
Kepler's third law: The area of an elliptical orbit is $A = \pi ab$. The area swept out by the motion $r(\phi)$ is $dA = (1/2)\vec{r}\cdot d\vec{r}$ with $d\vec{r} = rd\phi = r\dot{\phi}dt$ so that $\dot{A} = (1/2)r^2\dot{\phi} = l/2\mu$. This gives the period

$$\tau = \frac{A}{\dot{A}} = 2\pi\sqrt{\frac{\mu}{k}}a^{3/2} = \frac{2\pi}{\sqrt{G(M_1+M_2)}}a^{3/2}\,, \tag{33}$$

where we have recalled that $k = GM_1M_2$ and $\mu = M_1M_2/(M_1+M_2)$, and made use of the expressions for the semi-major axis a and semi-minor axis b, above.

For a planet with mass much smaller than the sun (good for the earth, less so for Jupiter) $M_1 + M_2 \simeq M_{\rm sun}$, in which case $\tau^2 \propto a^3$ for all such planets. This is Kepler's third law. (And in that case, the sun is at the focus; else, both the sun and the planet orbit the focus, on opposite sides).

Elliptical orbit: For bound orbits, the shape is an ellipse with the origin of $\vec{r}$ (i.e. the position of the center of mass, which is the position of the sun for light planets) at the focus. You should be aware of the following geometrical properties:

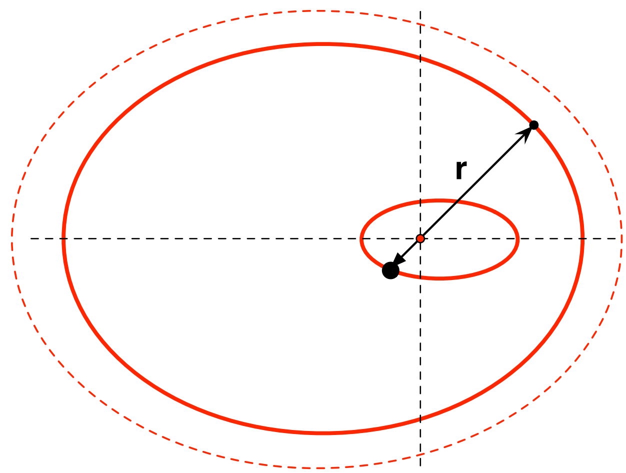


Elliptical orbit for attractive $1/r$ potential

For the case of a planet orbiting the sun, the extreme points of the major (x) axis, at r_{min} and r_{max} (the *classical turning points*), are called the *perihelion* (closest point to the sun) and *aphelion* (farthest point). For other stars, these are called the *periastron* and *apastron*. More generally, they are called the *apsides*.

Planet and Sun: This discussion is for the separation vector between the two bodies. The individual planet and sun position vectors from the center of mass are

$$\vec{r}_1 = \vec{R}_{\rm cm} + \frac{M_2}{M_2 + M_1}\vec{r}, \qquad \vec{r}_2 = \vec{R}_{\rm cm} - \frac{M_1}{M_2 + M_1}\vec{r} \tag{34}$$



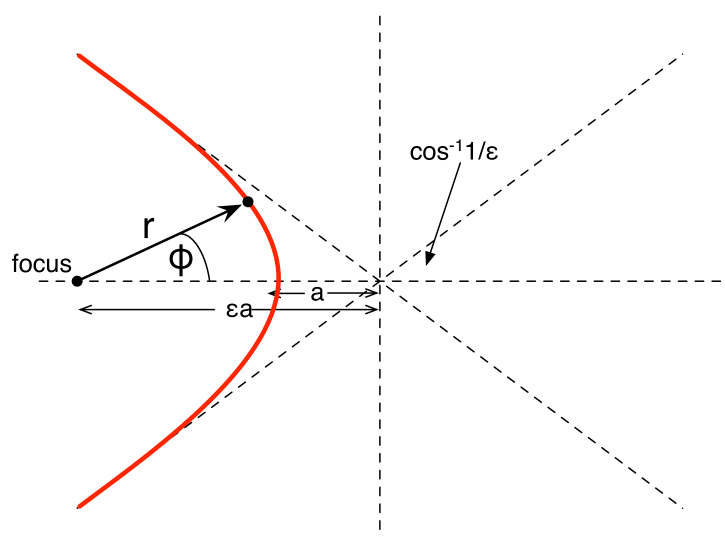
Planet and sun orbits: both rotate in elliptical orbits about the center of mass

Kepler's Laws revisited:

1. Planetary orbits are ellipses, with the center of mass at the focus
2. Conservation of angular momentum: the line joining the planets to the focus sweeps out equal areas in equal times: $\dot{A} = \frac{l}{2\mu}$.
3. The square of the period of the planet is approximately proportional to the cube of its mean distance to the Sun

$$\tau = \frac{2\pi}{\sqrt{G(M_{\rm s} + M_{\rm p})}} a^{3/2} \ .$$

Hyperbolic orbit: For unbound orbits, the shape is a hyperbola with the origin of $\vec{r}$ at the focus. The corresponding geometrical properties are:



Hyperbolic orbit for attractive $1/r$ potential

Here, the total energy $E = T + V = |k|/2a > 0$; the positive energy means the orbit is *unbound* (extends to $r \to \infty$).

Here, the red orbit passes "behind" the sun (at or near the focus), characteristic of an attractive potential (gravity, Keplerian motion). There is another hyperbolic orbit on the right side (mirror image), characteristic of a repulsive potential $V = +|k|/r$ (Coulombic, Rutherford scattering). This will be the subject of the next lecture. For repulsive potentials, only unbound hyperbolic orbits are possible.

Virial theorem

The virial theorem involves averages of the motion for interacting bound particles. It is most interesting for particles interacting with a power-law pair potential $V(r) \propto r^{\alpha}$, when the theorem takes the form

$$\langle T \rangle = \frac{\alpha}{2} \langle V \rangle \tag{35}$$

where the $\langle \rangle$ denotes an average over time. The average is taken over a period for periodic motion, or otherwise over a very long time in which case the result relies on the coordinates and momenta remaining bounded. In either case the method works only for bound states, not for scattering states which are nonperiodic and unbounded.

We have seen that for the SHO, with $\alpha = 2$, we have $\langle T \rangle = \langle V \rangle$ (this is a very special relationship!). For the Kepler problem, with the zero of the potential chosen to be $V(r = \infty) = 0$, we have

$$\langle T \rangle = -\frac{1}{2} \langle V \rangle \tag{36}$$

so that

$$E = \langle T \rangle + \langle V \rangle = \frac{1}{2} \langle V \rangle = -\langle T \rangle \ . \tag{37}$$

An important application of the virial theorem was by Zwicky who analyzed the Coma nebulae cluster, and found that the average kinetic energy implied more potential energy than could be supplied by the visible sources. Using this, he argued for the existence of dark matter. The work was done in 1933, although a more accessible reference is Astrophysical Journal **86**, 217 (1937).