Physics 106a, Classical Mechanics — Fall Term, 2019
Final Exam

Do not read beyond this page until you are ready to start the exam!

Due date: Due by **7pm in Physics 106 In Box in East Bridge on Friday 13 December**. Late exams will require extenuating circumstances; otherwise, no credit will be given. The "one free extension" *cannot* be used on the final.

Material: The exam covers the material covered in Lectures 1-19.

Logistics: The exam consists of this page plus two pages of questions. Do not look at the questions until you are ready to start the exam. Please use a blue book for the exam. Alternatively, you may type your solutions on individual sheets, but the typing time must be included in the allocated time. Please review the "Honor Code and Collaboration policy" on the course website.

The exam consists of five questions, with points as indicated.
Note that many of the problems can be done by a number of different methods: if possible check your results using a different approach. And check your answers make sense physically!

Time: you may spend up to 5 hours on the exam. You may take one or two breaks, as long as you not work on the exam or consult any sources that may relate to the exam during the breaks.

References: You may consult the class text books: Hand and Finch; and Taylor. You may also consult any material on the Ph106 website for this year, including lecture notes and solutions, and notes on this class you or another student in the class have made. You may not consult any other material, including other books, any material from previous years' classes or any other classes, or any other material on the internet. You may use a calculator but not any symbolic manipulation programs (you shouldn't need them). You may use a word processor to type your solutions if you wish.

If these instruction are not clear, please consult with me before starting the exam.

Grading: The exam will count for $30\% - 35\%$ of your grade for the class.

1. **Repulsive $1/r^3$ scattering**: A fixed (infinitely massive) force center scatters a particle of mass m and initial velocity v_0 according to the force law $F(r) = k/r^3$, with $k > 0$.

   (a) (1 points) Relate the impact parameter b to the magnitude of the total angular momentum l.

   (b) (2 points) Write down an expression for the total energy, in terms of $u(\theta) = 1/r(\theta)$, and l. (For scattering problems, it makes more sense to call the polar scattering angle θ rather than thinking of it as an azimuthal angle ϕ; do you understand why?).

   (c) (2 points) Solve for $u(\theta)$.

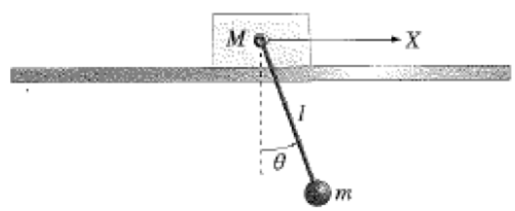
   (d) (2 points) Long after the collision, $r \to \infty$ or $u \to 0$, and $\theta \to \theta_\infty$. From this, derive $b(\theta_\infty)$.

   (e) (3 points) Show that the differential scattering cross section (with $\theta = \theta_\infty$) is

   $$\frac{d\sigma}{d\Omega} = \frac{k\pi^2(\pi - \theta)}{mv_0^2\theta^2(2\pi - \theta)^2 \sin\theta}.$$

   (You might want to plot it, but not for any extra credit).

2. **Small Oscillations and normal modes**
   A simple pendulum of mass m and length l is attached to a block of mass M that is free to move without friction along a horizontal track. As shown in the figure, there are two degrees of freedom, x and θ. (If your m's and M's look too much alike, call them m_p and m_b, respectively.)



   (a) (2 points) Write down the Lagrangian in both full generality, and then in the limit of small amplitude motion.

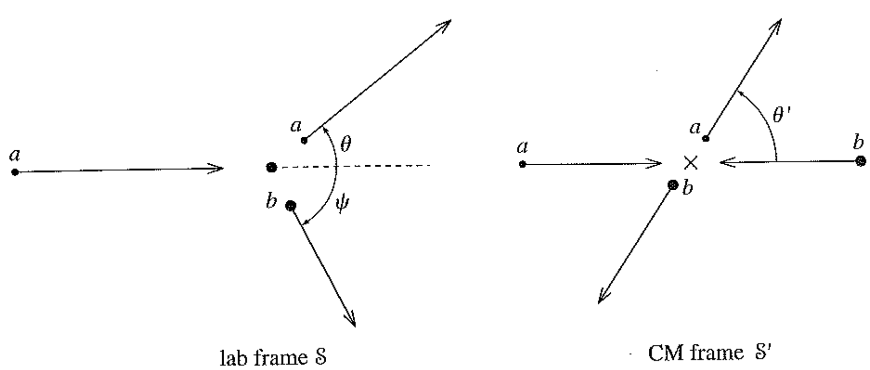
   (b) (1 points) Find the linearized equations of motion.

   (c) (3 points) Find the normal mode frequencies.

   (d) (3 points) Find the normal mode vectors of the system.

   (e) (1 points) Discuss the physical interpretation of the normal mode frequencies and eigenvectors.

3. **Geometry of elastic collisions:** Consider the relativistic collision shown in the figure:



   (CM is the center of momentum frame). In the lab frame the incoming particle has energy E_a and momentum $\vec{p}_a$.

(a) (3 points). Show that the velocity of the CM frame S' relative to the lab frame S is

$$\vec{V} = \frac{\vec{p}_a}{E_a + m_b}$$

(b) (3 points). By transforming between the two frames, show that

$$\tan\theta = \frac{\sin\theta'}{\gamma_V(\cos\theta' + V/v'_a)}$$

with v'_a the speed of a in the CM frame, and $\gamma_V = (1 - V^2)^{-1/2}$.

Now specialize to the case of $m_a = m_b$.

(c) (3 points). Show that in this case $V/v'_a = 1$, and find an expression for $\tan\psi$ analogous to the one above for $\tan\theta$.

(d) (3 points). Show that the angle between the two outgoing momenta is

$$\tan(\theta + \psi) = \frac{2}{\gamma_V V^2 \sin\theta'}$$

and that in the nonrelativistic limit of all speeds small compared with the speed of light this reduces to the well known result for the elastic collision of equal mass particles $\theta + \psi = 90°$.

4. **Pion production from the microwave background:** Consider the reaction between a nucleon N and the background microwave radiation approximated as a sea of photons with temperature $T = 3$K and typical energy $E_\gamma = k_B T \simeq 2.5 \times 10^{-10}$ MeV.

$$\gamma + N \to N + \pi$$

with the neutron mass-energy $m_N = 940$ MeV/c^2, and pion mass $m_\pi = 140$ MeV/c^2 (but use $c = 1$ here).

(a) (4 points). Using 4-vectors, find an equation giving the threshold energy E_N of the nucleon for this reaction to occur in terms of the given parameters, assuming a head on collision (photon and nucleon moving in opposite directions).

(b) (1 point). Evaluate E_N in MeV to one significant figure using the approximation $E_N \gg m_N$.

5. **Pion decay**: A positive pion (mass m_π) decays into a positive muon (mass m_μ) and a neutrino (essentially zero mass), $\pi^+ \to \mu^+ + \nu$.

(a) (3 points). If the pion is at rest, compute the total energy, kinetic energy, momentum, and speed, of the outgoing muon. (Simplify all the expressions!) (If you are curious, you may use a calculator to compute what fraction of the speed of light this is, using $m_\pi = 140$ MeV and $m_\mu = 106$ MeV.)

(b) (2 points). If the pion is moving with a total energy of E_π MeV, what are the minimum and maximum energies possible for the outgoing muon?