Physics 106a: Assignment 7

22 November, 2019; due 7pm Friday, 6 December (**TWO WEEKS**) in the "Ph106 In Box" in East Bridge mailbox. The final exam will be posted on Friday, 6 December; due Friday, 13 December.

Hamiltonian Formulation, Special Relativity

Reading

Hamiltonian mechanics: Hand and Finch Chapter 5. Special Relativity: Hand and Finch Chapter 12.

Problems

1. **Spherical pendulum:** This is the standard pendulum of a mass m on a string of length l in gravity g but considering the full range of motion on the surface of the sphere of radius l assuming the string remains taut. Use angles θ of the string to the vertical, with $\theta = 0$ the rest state, and ϕ the angle about the vertical. Hand and Finch treat this problem using the Hamiltonian in Example 3 of §5.5 (I use l instead of R for the length of the string). You will use the Routhian approach.

   (a) Preliminaries: write down the Lagrangian and hence the momenta p_θ, p_ϕ conjugate to θ, ϕ; show that the coordinate ϕ is ignorable (cyclic).

   (b) Find the Routhian $\mathcal{R}(\theta, \dot{\theta}, \phi, p_\phi)$. (Actually, of course, there will be no dependence on ϕ).

   (c) Using $\mathcal{R}$ as the Lagrangian for the θ motion, show directly that the equation of motion can be written in the form
   $$ml^2 \frac{d^2\theta}{dt^2} = -\frac{d}{d\theta} V_{\text{eff}}(\theta),$$
   and find the effective potential $V_{\text{eff}}(\theta)$, which will depend on the constant value of p_ϕ.

   (d) For motion in a vertical plane $p_\phi = 0$ we know there are oscillating solutions about $\theta = 0$ and running solutions where the pendulum passes through $\theta = \pi$ and continues to rotate in one direction. Show for $p_\phi \neq 0$ the θ motion is bounded $0 < \theta < \pi$ and is periodic.

   (e) Show that conical motion $\theta = \theta_0 =$ constant, $p_\phi \neq 0$ only occurs for $\theta_0 < \pi/2$, i.e., for the string at an angle lower than horizontal.

   (f) Find the frequency ω of the small amplitude oscillations of θ about θ_0 for $p_\phi \neq 0$ in terms of g, l, θ_0, and the ratio $\omega/\dot{\phi}$ in terms of just θ_0) and find the limits for $\theta_0 \to 0$ and $\theta_0 \to \pi/2$. Reconcile your answer for $\theta_0 \to 0$ with what we expect for the small amplitude motion using x, y coordinates in the horizontal plane, namely periodic motion with frequency $\sqrt{g/l}$. A sketch of the motion in the x, y plane implied by the θ, ϕ motion may help. Give a physical description of the oscillations for $\theta_0 \to \pi/2$, including a simple argument for $\omega/\dot{\phi}$ for the motion in this limit.

2. **Poisson Brackets with angular momentum:** (H&F problem 6.15).

   (a) Calculate all the Poisson brackets of the components of $\vec{r}$ and $\vec{p}$ with each other (for example, $[x, y]$, $[x, p_x]$, $[x, p_y]$, etc.).

   (b) Calculate all the Poisson brackets of the components of $\vec{r}$ and $\vec{p}$ with the components of the angular momentum $\vec{l} = \vec{r} \times \vec{p}$ (for example, $[x, l_x]$, $[x, l_y]$, $[p_x, l_x]$, etc.). You may find it convenient to use the Levi-Civita completely antisymmetric tensor ε_{ijk} (look it up) such that $l_i = \varepsilon_{ijk} r_j p_k$.

   (c) Prove that $[l_x, l_y] = l_z$ for all cyclic permutations of l_x, l_y, l_z.

3. **Poisson Brackets with constants of motion:** (H&F problem 6.14).

   Consider the uniform motion of a free particle of mass m. The Hamiltonian is a constant of the motion, and so is the quantity $F(x, p, t) \equiv x - pt/m$.

   (a) Compare $[H, F]$ with $\frac{\partial F}{\partial t}$. Prove that F is also a constant of the motion.

(b) Prove that the Poisson bracket of two constants of the motion is itself a constant of the motion, even if the constants $F(x, p, t)$ and $G(x, p, t)$ depend explicitly on the time. (Part a is one example of this).

(c) Show *in general* that if the Hamiltonian and a quantity F are constants of the motion then $\frac{\partial F}{\partial t}$ is a constant of the motion also.

4. **Light, a bullet, and a ruler:** In frame S, a ray of light and the path of a bullet with speed u run along the edge of a stationary ruler at an angle θ to the x-axis in the xy plane. Now look at the angles θ' relative to the x'-axis observed in the frame S' moving with speed v along the x-axis of S.

   (a) What is the angle θ'_r that the ruler is measured to have?

   (b) What is the angle θ'_b of the path of the bullet?

   (c) What is the angle θ'_l of the light ray?

   (d) Are the three angles $\theta'_r, \theta'_b, \theta'_l$ equal? Do the bullet and light ray run along the edge of the ruler in the S' frame? Explain your answer in terms of frame independent events.

5. **Addition of velocities:** A particle has velocity $\vec{u}' = (u'_x, u'_y, u'_z)$ in the frame S' moving at speed v along the x-axis relative to frame S (our standard configuration). Derive yourself using $u'_x = dx'/dt'$ and $u_x = dx/dt$ etc. and the Lorenz transformations for dx, dt (but you need not hand this part in), or find in a textbook, the expressions for the components of the velocity $\vec{u}$ in S

$$u_x = \frac{u'_x + v}{1 + u'_x v}, \qquad u_y = \frac{u'_y}{\gamma_v (1 + u'_x v)}, \qquad u_z = \frac{u'_z}{\gamma_v (1 + u'_x v)},$$

   with $\gamma_v = (1 - v^2)^{-1/2}$;

   A stick is measured to be length L in its rest frame S. In a frame S' moving at speed $\frac{3}{5}$ along the $+x$ axis of S the stick is seen to move by at a speed $\frac{3}{5}$ along the $-x'$ direction. In the frame S' a ball also moves by at the same speed but in the $+x'$ direction.

   (a) In the frame S', I calculate the time $\Delta t'$ it takes the ball to pass the stick as the contracted length L/γ with $\gamma = \frac{5}{4}$ divided by the relative speed $\frac{6}{5}$. Is this the correct answer? Is it consistent with the idea that "nothing travels faster than the speed of light"?

   (b) Find the speed of the ball in the rest frame of the stick, and the speed of the stick in the rest frame of the ball. What is the length of the stick measured in the frame S'' moving with the ball?

   (c) From part (b) find the time $\Delta t''$ measured in the frame of reference moving with the ball between the two ends of the stick passing the ball. Show that this is consistent with the time measured in S' evaluated in part (a) and the Lorenz transformation between the two frames S' and S'' of the coordinates for the two events: ball coincides with one end of stick; ball coincides with other end of stick.