Physics 106a: Assignment 5

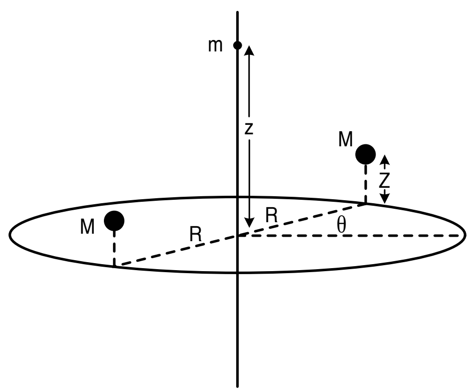
8 November, 2019; due 5pm Friday, 15 November in the “Ph106 In Box” in East Bridge mailbox.
NOTE: This assignment is way to long, so leave out Problem 3.

One Dimensional Systems: Central Forces and the Kepler Problem

Reading: Hand and Finch Chapter 4.

Problems

1. **Binary-sun solar system**: Consider a binary pair of identical suns of mass M orbiting in the $x-y$ plane in an orbit centered at the origin. The gravitational constant is G. Now add a planet of small mass m with an initial condition on the z axis above the center of mass of the two suns and with a velocity in the z direction; by the symmetry of the system the small mass will remain on the z axis and suns will have equal z coordinates and their center of mass will also remain on the z axis. We can choose as coordinates describing the dynamics: the z coordinate of the planet z; the z coordinate of the two suns Z; and the polar coordinates (R, θ) giving the x, y coordinates of the suns $(\pm R\cos\theta, \pm R\sin\theta)$ — see figure.



   (a) What is the Lagrangian of the system in terms of these coordinates and their time derivatives?

   (b) Using constants of the motion implied by translational and rotational symmetries find the coupled equations of motion for $\ddot{q}, \ddot{R}$ with $q = z - Z$. Your result will depend on the angular momentum l of the two suns as well as the parameters of the problem.

   (c) In the limit of small planetary mass $m \ll M$ we can ignore the effect of the planet on the motion of the suns. Find the explicit solution for the motion of the suns $R(t)$ for orbits with small eccentricity ϵ. Write your solution as circular motion plus a term proportional to ϵ.

   (d) Hence show for these limits, keeping terms linear in ϵ, that the equation of motion for q is a nonlinear oscillator (oscillator with a non-quadratic potential) driven by a term $\epsilon f(q)\cos\Omega t$ with $2\pi/\Omega$ the period of the solar motion, and find the form of $f(q)$.

   For your interest only: The dynamics of this system given by the equation of motion you have calculated turns out to be complex even in the sensible limit $m \ll M$. One way of illustrating the complexity is to consider the crossing times τ_1,$\tau_2 \ldots$ for which the planet crosses the plane of the orbit of the suns. It can be shown that for *any* chosen sequence of numbers (e.g. random numbers) an initial condition can be found such that the τ_i reproduce this sequence, with escape to infinity corresponding to the end of a finite sequence. Thus no matter how many τ_i are measured from observation of such a system, no prediction can be made for the next τ in the dynamics! These chaotic dynamics are a demonstration of the difficulty of analyzing the 3 body problem in gravitational physics.

2. **The Yukawa potential:** A particle of mass m moves in a force field described by the Yukawa potential $V(r) = -\frac{k}{r}\exp(-\frac{r}{a})$, where k and a are positive. In the limit $a \to \infty$, this reduces to the Kepler problem. When a is finite, this results in an attenuation of the potential, and force, for $r \gg a$;

i.e., a *short-range* potential. It is commonly used to describe the potential between nucleons in the nucleus, with $a \approx 10^{-15}$ m.

(a) Write the equations of motion and reduce them to the equivalent one-dimensional problem, in terms of an effective potential.

(b) Use the effective potential to identify and discuss the qualitative nature of orbits for different values of the energy E and the angular momentum ℓ.

(c) In the limit $a \to \infty$ (the Kepler problem), give expressions for the radius of the circular orbit r_c, the orbital frequency (inverse of the orbital period), and the energy, in terms of ℓ. k, and m.

(d) For the Yukawa potential, give the equation whose solution yields the radius of the circular orbit $r_{c,a}$ in terms of ℓ. k, m and a.

(e) In the limit $r_c/a \ll 1$, solve the equation from part (d) by approximation, to give $r_{c,a}$ in terms of r_c and a. You can assume $r_{c,a} = r_c g$, where g is a function of r_c/a (and which should go to 1 in the limit $a \to \infty$). Find g to leading order in r_c/a. Also find the orbital frequency in this case.

(f) If the orbit is nearly circular, i.e. $r(t) = r_{c,a} + \delta r(t)$ ($\delta r(t) \ll r_0$), derive the differential equation for $\delta r(t)$. Show that it is stable, and corresponds to oscillations about the circular orbit, and find the frequency of oscillation. compare to the orbital frequency.

(g) Think about how you might use observations of planetary motion to measure or limit the value of a. You might also want to think about what physics might result in a Yukawa potential.

3. **Precessing ellipses**: Hand and Finch Problem 4-24 (slightly changed). **This problem is NOT required for A5.** Discuss the motion of a particle in a central force potential

$$V(r) = -\frac{k}{r} + \frac{\beta}{r^2} .$$

In particular, show that the equation of the orbit has an exact solution in the form

$$\frac{p}{r} = 1 + \epsilon \cos \alpha\phi .$$

This is an ellipse for $\alpha = 1$, but it is a *precessing* ellipse if $\alpha \neq 1$. The precessing motion may be described in terms of the rate of precession of the apsides (turning points). Derive an approximate expression for the rate of precession when α is close to 1. Calculate the precession angle for one period of the (almost) elliptical motion. If β is increased to the point where it is no longer small compared to the centrifugal term, how does this affect the orbit?

4. **Hyperbolic orbits - asymptotes and impact parameters** Hand and Finch Problem 4-27.

A two-body system with reduced mass μ and orbital angular momentum l in a potential $V(r) = \pm k/r$ with $k > 0$, with eccentricity $\epsilon > 1$, the (unbound) orbit is described in Cartesian coordinates as:

$$p = -\sqrt{x^2 + y^2} + \epsilon x ,$$

where $p \equiv l^2/(\mu k)$ is fixed by the initial conditions. The asymptotes (for both attractive and repulsive forces) are straight lines.

(a) Prove that the equations for those lines in Cartesian coordinates for a repulsive force will have the general form

$$y = \pm \left(\sqrt{\epsilon^2 - 1}\right) x - \frac{\epsilon p}{\sqrt{\epsilon^2 - 1}} .$$

The asymptotes for an attractive inverse-square force obey a similar equation, but with $\epsilon \to -\epsilon$.

(b) The *impact parameter* b is defined as the distance of closest approach to the origin along the incoming asymptote. Prove that the angular momentum l is given in terms of the reduced mass μ and the center of mass relative velocity v_∞ infiniely far away from the origin by the formula $l = \mu v_\infty b$.

(c) Show that the impact parameter can be written as a function of ϵ for constant total energy in the center of mass, E:

$$b(\epsilon) = \frac{k}{2E}\sqrt{\epsilon^2 - 1} .$$