Physics 106a: Assignment 4

1 November, 2019; due 7pm Friday, 8 November in the "Ph106 In Box" in East Bridge mailbox.

Oscillations and Normal Modes

Reading

Hand and Finch Chapters 3 and 9.

Problems

1. **Driven critically damped oscillator:** (This is Hand and Finch Problem 3-21). A critically damped oscillator has $Q = 1/2$. The free oscillator obeys the homogeneous equation of motion $\ddot{q} + 2\dot{q} + q = 0$ (in natural units where $\omega_0 = 1$, or equivalently, using rescaled dimensionless time $\omega_0 t \to t$). The two free oscillator solutions are e^{-t} and te^{-t}.

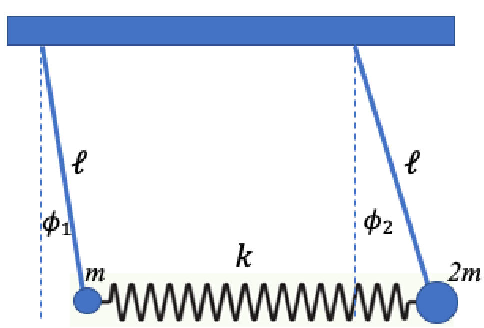
   (a) Drive this oscillator with an external driving force that is a discontinuous step at $t = 0$: for $t < 0$, $F = 0$, and for $t \geq 0$, $F = 1$. Assuming that $q = \dot{q} = 0$ for $t < 0$, find an explicit solution for $t \geq 0$.

   (b) Consider the oscillatory driving force: for $t < 0$, $F = 0$, and for $t \geq 0$, $F = \cos t$. Again, $q = \dot{q} = 0$ for $t < 0$. Find the form of the steady-state ($t \gg 1$) solution by first solving for $q(t)$ for the complex driving force $F = e^{it}$, $t > 0$, and then finding the physical displacement of the oscillator $q(t)$ for $F(t) = \cos t$. What is the relative phase between the driving force and the oscillator response in the steady state?

   (c) To find the exact solution for all positive times you could use a Green's function or you could match boundary conditions at $t = 0$. Use the boundary condition method to find the transient solution. Combine this with the result of part (b) to find the oscillator's total response to suddenly turning on a $\cos t$ driving force at $t = 0$. Make a sketch of $q(t)$ for $0 \leq t \leq 5$ (or more). For what time is the response maximized? (You may want to use Mathematica or other software).

   (d) The derivative of a step function is a delta function. From this fact, find the response of this oscillator to a delta function impulse at $t = 0$. Then find the explicit form of the Green's function $G(t - t')$. Write the oscillator response to the driving force in part (b), as an integral over t'. What are the limits of integration? It is easy enough to do the Green's function integral, e.g. using Mathematica, so you might want to do this, for no credit, to check the result in part (c).

2. **Coupled pendulums:** Two pendulums of identical length l with masses m and $2m$ are connected by a spring of spring constant $k = 2mg/l$. Use the variables ϕ_1 for the lighter mass and ϕ_2 for the heavier mass. The spring is unstretched when the two pendulums are vertical $\phi_1 = \phi_2 = 0$.



   (a) What is the Lagrangian up to quadratic order in the small displacement and velocities?

   (b) Find the kinetic and potential energy matrices.

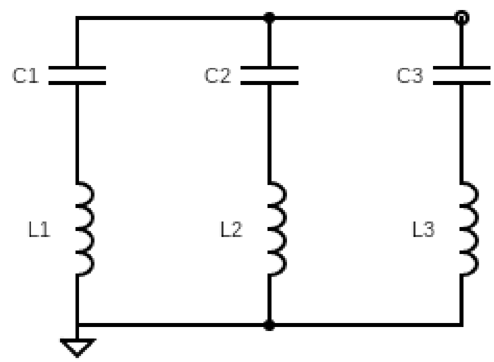
   (c) Using these, show that the normal mode frequencies are ω_0 and $2\omega_0$ with $\omega_0 = \sqrt{g/l}$.

(d) For an initial condition of both pendulum released from rest at $t = 0$, find the ratio of the displacements $\phi_1(0)/\phi_2(0)$ such that only the higher frequency oscillation is excited.

(e) A small horizontal force $F = F_0 \cos \omega t$ is applied to the lighter mass. Find an expression for the displacement $\phi_1(t)$ in the driven motion for general ω. You may assume the transient response at the resonant frequencies has died out due to some very small damping.

3. **Coupled LC oscillators:**

Find expressions for the eigenfrequencies of the following electrical coupled circuit:



4. **Triple coupled pendulums:** (H&F problem 3.2.) A "triple pendulum" (three pendulums each coupled to the adjacent two via springs) has kinetic and potential energies given by

$$T = \frac{1}{2}\left(\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2\right)$$

$$V = \frac{1}{2}\left(\theta_1^2 + \theta_2^2 + \theta_3^2 - 2\epsilon\left[\theta_1\theta_2 + \theta_2\theta_3 + \theta_3\theta_1\right]\right)$$

respectively, where m, g and l have been set equal to one for convenience, and ϵ contains the spring constant. (Don't try to picture this!)

(a) What are the T and V matrices?

(b) Find the mode frequencies.

(c) Are there any restrictions on the value of the constant ϵ?

(d) Show that a particular set of orthonormal mode eigenvectors can be chosen to be, in the coupled basis of $(\theta_1, \theta_2, \theta_3)$,

$$x_1 = \sqrt{\frac{2}{3}}\,(1,1,1)\,,$$
$$x_2 = (1,0,-1)\,,$$
$$x_3 = \sqrt{\frac{1}{3}}\,(1,-2,1)\,,$$

and associate the frequencies from above with each mode.

(e) Draw these modes.