Physics 106a: Assignment 3

18 October, 2019; due 7pm Friday, 25 October in the "Ph106 In Box" in East Bridge mailbox.

Constrained dynamics and small oscillations

Reading

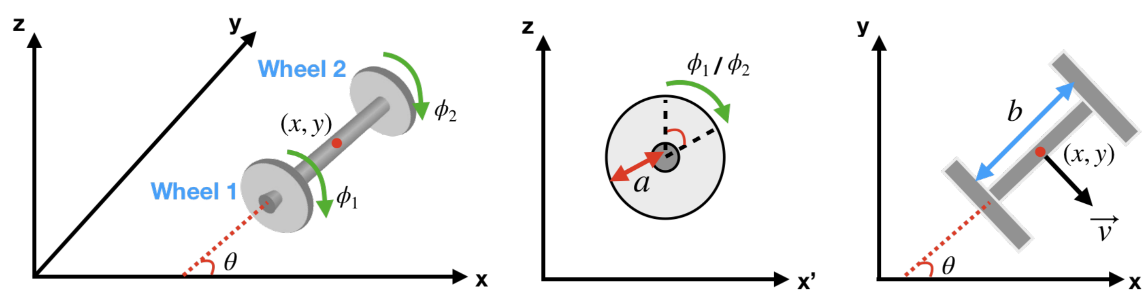
Constrained dynamics: HF §2.6-8, T §7.3,7.10. You should work through some problems on the simpler use of Lagrange multipliers in the constrained optimization of functions (rather than functionals, as used in Hamilton's principle) if you find this approach confusing.

Oscillations: I'm presuming you have seen much of this material before. You should understand all of the topics discussed in the notes to Lecture 5 You can review the material in HF Ch 3 or T Ch5.

Problems

Problem 3 is longer and will count for twice as much as the other problems.

1. **Holonomic and nonholonomic:** Two wheels of radius a are mounted on the ends of a common axle of length b. The wheels can rotate independently. The whole combination rolls without slipping on a flat table. If ϕ_1, ϕ_2 are the rotation angles of the two wheels on the axle, θ is the angle between the axle and a fixed line, and (x, y) are the coordinates of the center of the axle, find *three* constraints between the coordinates and say whether each one is holonomic or nonholonomic. You should find one holonomic constraint and two nonholonomic constraints.



3D view of the system   Side view of the wheel   Top view of the system

2. **Particle constrained to elliptical wire (revisited):** In this problem you find the equations of motion for the bead of mass m on a frictionless elliptical wire in a vertical plane with gravity g discussed in Assignment 1, Problem 5, but now using the method of Lagrange multipliers. The horizontal semi-axis is a and the vertical semi-axis b, assumed to be time independent.

   (a) Find the Lagrangian $L(x, y, \dot{x}, \dot{y})$ using the Cartesian coordinates (x, y) with x horizontal and y vertical.

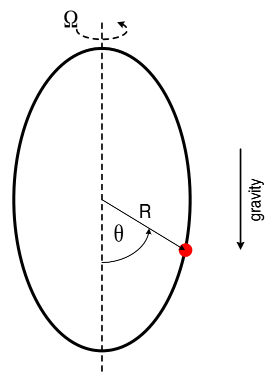
   (b) Use Hamilton's principle to find the equations of motion for x and y, introducing the constraint that the bead is on the ellipse using the method of Lagrange multipliers.

(c) *After* you have found the equations of motion, introduce the representation $x = a\cos\alpha$, $y = b\sin\alpha$ that enforces the constraint and eliminate the Lagrange multiplier to find the equation of motion for α. Check against the equation of motion derived directly using the generalized coordinate α in Problem 5 of Assignment 1.

(d) Find the constraint force in terms of the Lagrange multiplier and verify from this expression that it is perpendicular to the wire.

3. **Bead on rotating hoop:** This problem reviews much of the material covered so far.

A bead of mass m moves without friction on a circular hoop of radius R which rotates at a fixed angular frequency Ω about the vertical z axis in gravity g, as shown in the figure.



Answer the following:

(a) Pick out the correct word(s) describing the constraint (that the wire remain on the bead) from the following list: holonomic, nonholonomic, time dependent (rheonomic), time independent (scleronomic).

(b) The constrained dynamics has one degree of freedom, that may be described by the generalized coordinate θ. Find the generalized force $\mathcal{F}_\theta$ when the mass is at angle θ.

(c) Derive expressions for the kinetic energy T and potential energy V, and show that these lead to the expression for the Lagrangian L (up to constants etc.)

$$L = \frac{1}{2}mR^2\dot{\theta}^2 + \frac{1}{2}mR^2\sin^2\theta\,\Omega^2 + mgR\cos\theta$$

Use this expression for the Lagrangian to answer the following:

(d) Find the equation of motion.

(e) What is the Hamiltonian as a function of $\theta, \dot{\theta}$, and is it a constant of the motion?

(f) Is the total energy $T + V$ a constant of the motion? If not, where does the energy come from to change the total energy (potential + kinetic) of the bead?

(g) Show that $\theta = 0$ and $\theta = \pi$ are equilibria (θ is time independent) for all rotation rates. Are the equilibria stable or unstable? Does this change as a function of Ω?

(h) Show that for rotation rates Ω greater than a critical rotation rate Ω_c an additional pair of equilibria develop at $\pm\theta_r$ and find θ_r as a function of g, R, and Ω.

(i) Is the new solution pair stable? If so, find the frequency of linear oscillations about them.

Finally:

(j) Write down an effective action that would allow you to solve the problem using Cartesian coordinates x, y, z and the method of Lagrange multipliers.

4. **Particle in a bowl:** (Hand & Finch 3.1)

   A point particle of mass m is confined to the frictionless surface of a half-spherical bowl, in the presence of a uniform gravitational acceleration g. There are two degrees of freedom.

   (a) Prove that the equilibrium point is the bottom of the bowl.

   (b) Does the bowl have to be exactly spherical for this to be true? Near to the bottom of the bowl, what is the most general form possible for the shape of the bowl in order to maintain the stability of the equilibrium point at the bottom?

   In the general case, with a non-spherical bowl of arbitrary shape, no analytical solution is known for the motion of the point mass; but one can solve the equations of motion numerically (in simulation).