Physics 106a: Assignment 2

11 October, 2019; due 7pm Friday, 18 October in the "Ph106 In Box" in East Bridge mailbox.

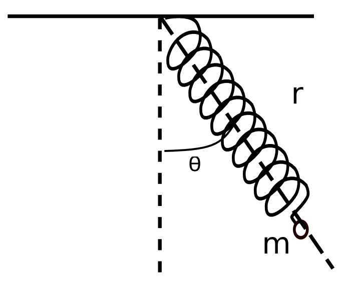
Constraints, Virtual Work, and Lagrangian Dynamics

Reading

You can now read the complete chapters 1 and 2 in Hand and Finch or 6 and 7 in Taylor on Lagrangian mechanics and the variational approach.

Problems

1. **Hanging spring:**



A massless spring with rest length l_0 (with no tension) and stiffness k hangs in the gravity field. It has a point mass m connected to one end and the other end fixed. The motion of the system is only in one vertical plane (the plane of the paper).

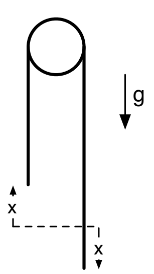
   (a) Write down the Lagrangian.

   (b) Find Lagrange's equations using variables θ, $\lambda = (r - r_0)/r_0$, where r_0 is the rest length (hanging with mass m). Use $\omega_s^2 = k/m$, $\omega_p^2 = g/r_0$ to write your results.

   (c) Discuss the lowest order approximation to the motion when λ and θ are small with the initial conditions $\theta = 0$, $\dot{\lambda} = 0$, $\lambda = A$, $\dot{\theta} = \omega_p B$ at $t = 0$. A and B are constants.

   (d) Discuss the next order approximation to the equation of motion for λ. Under what conditions will the λ motion resonate? Can this be realized physically? (Hint: you may use the lowest order results for θ motion.)

2. **Massive rope and pulley, frictional forces:** A rope of length l and mass ρ per unit length runs over a frictionless massless pulley of radius $R < l/\pi$, initially with equal lengths on either side.



The system is immersed in oil of negligible density so that the only effect of the oil is to act on a piece of rope moving with speed v with a force per unit length of magnitude γv in the opposite direction to the motion. One end of the rope is given a small initial displacement downwards, and then gravity g acts to accelerate the rope so that it slides over the pulley. The problem is to understand the motion, which will be described by the generalized coordinate x measuring the downward displacement of one end of the rope (choose the end that moves downwards).

   (a) What is the virtual work corresponding to a virtual change δx in x, in terms of x and $\dot{x}$ and the parameters of the problem?

   (b) What is the generalized force $\mathcal{F}_x$ corresponding to the generalized coordinate x, in terms of x and $\dot{x}$ and the parameters of the problem?

(c) Use D'Alembert's principle or the generalized equation of motion (Golden Rule #1) to show that the equation of motion takes the form

$$\ddot{x} + \alpha\dot{x} + \beta x = 0$$

and find expressions for α, β in terms of the parameters of the problem.

(d) Check your answer by solving the case of $\gamma = 0$ using the Lagrangian approach. What is the solution $x(t)$ for $\gamma = 0$ for an initial condition $x = x_0, \dot{x} = 0$, for times such that the rope remains fully in contact with the pulley?

3. **The Hamiltonian - Quadratic forms** - this is Hand & Finch problem 1.9.

(a) Prove that, if the constraints are scleronomic (i.e., time-independent), the kinetic energy T is a quadratic function (*quadratic form*) of the generalized velocities. Then prove that this implies

$$\sum_k \dot{q}_k \frac{\partial T}{\partial \dot{q}_k} = 2T.$$

(b) Assuming that the kinetic energy T *is* a quadratic form of the generalized velocities so that the formula above is correct, and also that the potential V depends only on coordinates and not velocities, prove that the Hamiltonian

$$H = \sum_k \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L$$

is the total energy ($H = T + V = E$).

4. **Two ways to handle constraints:** (Note: parts (b) and (c) of this problem depend on material in lecture 5 on Lagrange multipliers.) A bead of mass m slides on a vertical frictionless wire with the shape $z = \sin x$ in gravity g. (The coordinate z is vertical and x horizontal.) Find the equation of motion for the dynamics in two ways:

(a) Use the variable x as the single generalized coordinate and evaluate the Lagrangian for the constrained motion in terms of $x, \dot{x}$. Now use the Euler-Lagrange equation to derive the equation of motion for x.

(b) Write the Lagrangian in terms of $x, \dot{x}, z, \dot{z}$ and find the equations of motion for x, z using Hamilton's principle, introducing the constraint using the method of Lagrange multipliers.

(c) Eliminate your Lagrange multiplier from part (b) and use the constraint equation to show that you get the same equation of motion as in part (a).