Physics 106a: Assignment 1

4 October, 2019; due 7pm Friday, 11 October in the "Ph106 In Box" in East Bridge mailbox.
Remember: only identify your solution set with your class id number, not your name, unless you signed a FERPA waiver.

Variational Approach and Lagrangian Dynamics

Reading

I am presenting the material on introducing Lagrangian dynamics in a different order than Hand and Finch. The material I have discussed in the first week is in §1.1 and §2.1-5. I will be filling in the gaps next week. The calculus of variations is used in many contexts: the textbooks discuss some of these, and others are treated in their problems. I give one example as Problem 4, and you should work through some other examples too.

I will not discuss a different way of formulating the variational approach to particle dynamics known as Maupertuis' Principle: this is discussed in the Appendix to Chapter 2 of Hand and Finch which you can read if you are interested.

Problems

1. **1-D Motion with potential:**

   Consider a 1-D motion, governed by the equation

   $$\frac{1}{2}\dot{y}^2 + V(y) = E \tag{1}$$

   with

   $$V(y) = \begin{cases} -y^2\,, & |y| < 1 \\ \\ -1\,, & |y| >= 1. \end{cases} \tag{2}$$

   (a) Sketch the potential energy V.

   (b) Suppose we start at rest at $y = \delta$ with $0 < \delta < 1$, compute the amount of time T it takes for the particle to reach $y = 1$. You may find a table of integrals useful for solving the differential equation.

   (c) Show that $T \to +\infty$ when $\delta \to 0$.

2. **Momentum and Angular Momentum:**

   A particle of mass m moves according to the equations

   $$\begin{aligned} x &= x_0 + a\,t^2 \\ y &= b\,t^3 \\ z &= c\,t \end{aligned}$$

   Find the angular momentum $\vec{L}$ at any time t. Find the force $\vec{F}$ and from it the torque $\vec{N}$ acting on the particle. Verify that the torque gives the rate of change of angular momentum.

3. **Conservative vs non-conservative forces:**

We explained in class that a conservative force is one for which the work done in moving from one point to another is path-independent. It is equivalent to say that the work done around a closed path vanishes. Stokes' theorem relates the line integral of a vector around a closed path to the integral of the curl of the vector over the area enclosed by that path:

$$\oint_C \vec{F} \cdot d\vec{r} = \int_{A_C} d\vec{a} \cdot \left(\vec{\nabla} \times \vec{F}\right) \tag{3}$$

Thus, any conservative force must have $\vec{\nabla} \times \vec{F} = 0$ everywhere to ensure the left side vanishes for an arbitrary loop C. Calculate the curl to determine which of the following force fields is conservative. For any that are conservative, find the potential energy $U(\vec{r})$.

(a) $F_x = a\,y\,z + b\,x + c$, $F_y = a\,x\,z + b\,z$, $F_z = a\,x\,y + b\,y$. Consider the entire 3-D space.

(b) $F_x = -z\,e^{-x}$, $F_y = \ln z$, $F_z = e^{-x} + \frac{y}{z}$. Consider only the region of $z > 0$.

(c) $F(\vec{r}) = \vec{h} \times \vec{r}$. Consider the entire 3-D space.

(d) $F_x = y/(x^2+y^2)$, $F_y = -x/(x^2+y^2)$, $F_z = 0$. [Extra: not for credit; it's tricky. Consider the 3-D space with the line of $x = y = 0$ removed]

where a, b, c, and $\vec{h}$ are constants.

4. **Calculus of variations and geodesics**

(a) By extremizing the length of a path $y(x)$ between two points on the (x, y) plane using the calculus of variations, show that the shortest path between two points (the geodesic) on a plane is a straight line. (You don't have to show it is the shortest, only that it is extremal.)

(b) For the geodesic on a sphere, use spherical polar coordinates θ, ϕ. Derive the expression for the length of a path $\phi(\theta)$ between two fixed points on the unit sphere. Use the calculus of variations to find a differential equation for ϕ that specifies the shortest path. Work the problem far enough to find a first order differential equation for $d\phi/d\theta = \ldots$ (Again, you don't have to show it is the shortest, only that it is extremal. This works for both the shortest *and* the longest paths; can you picture both?)

Extra (not for credit): It is somewhat tricky calculus/algebra to integrate the equation for $d\phi/d\theta$, and then to show the resulting expression defines a great circle — the intersection of the plane containing the two points and the center with the surface of the sphere. You might at least like to show that one solution is $d\phi/d\theta = 0$ i.e. $\phi =$ constant, the special case of a great circle through the two poles. The problem is harder to solve if we write the path as $\theta(\phi)$ rather than $\phi(\theta)$: do you see why this is so?

5. **Particle constrained to elliptical wire:** In this problem you use the Lagrangian approach to find the equation of motion for a particle of mass m sliding on a frictionless vertical elliptical wire, with horizontal principal axis a and vertical principal axis b, in gravity g. Use the angle α giving the Cartesian coordinates of the mass as $x(t) = a\cos\alpha(t), y(t) = b\sin\alpha(t)$ to describe the motion.

(a) Verify that with the given parametrization, the bead lies on the ellipse for all time, so that the constrained dynamics can be described by the single variable $\alpha(t)$.

(b) Now derive the differential equation of motion for α.

(c) You should find a term in the equation of motion that is proportional to $\dot{\alpha}^2$, as well as the "obvious" $\ddot{\alpha}$ acceleration term. Give an intuitive interpretation of the $\dot{\alpha}^2$ acceleration term, including why this term disappears in the circular limit $a = b$ and thinking of the constraint forces of the wire on the bead.